Two Dimensional Harmonic Maps into Lie Groups

by

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Thesis
Submitted to the Faculty of the Graduate School of
The Chinese University of Hong Kong
(Division of Mathematics)

In partial fulfillment of the requirements
for the Degree of
Master of Philosophy

July, 1999

摘要

我們概述了一部份 Uhlenbeck[12] 中得到的結果。在她的論文中，Uhlenbeck 定義得到了二維的到 U_n 上的調和映射的廣義解(extended solution)。而在廣義解上群的作用則誘導出在相對應的調和映射上的作用。由此，Uhlenbeck 得到一個對於映入 U_n 的調和映射的 Backlund 變換。在此文中，我們也考慮把 Uhlenbeck 的結果推廣到二維的映入緊李群的調和映射，且對此作一些綜述。

Abstract

We survey part of the results in Uhlenbeck[12]. In [12], Uhlenbeck defined the notion of extended solutions of 2-dimensional harmonic maps into U_n. A group action on extended solutions induces an action on the corresponding harmonic maps. From this, she defined a Bäcklund transformation of the harmonic map into U_n. We will also survey some generalization of Uhlenbeck's results to 2-dimensional harmonic map into a compact Lie group.

ACKNOWLEDGMENTS

I wish to express my gratitude to my supervisor Professor Wan Yau Heng Tom. Without his generous help and continuous encouragement, the writing of this thesis will not be possible.

Contents

1 **Introduction** 5

2 **Preliminary** 12

2.1 Lie Group and Lie Algebra 12

2.2 Harmonic Maps 15

2.3 Some Factorization theorems 17

3 **A Survey on Unlenbeck's Results** 22

3.1 Preliminary 24

3.2 Extended Solutions 26

3.3 The Variational Formulas for the Extended Solutions 30

3.4 The Representation of $\mathbb{A}(\mathbb{S}^2, G)$ on holomorphic maps $\mathbb{C}^* \to G$ 33

3.5 An Action of $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$ on extended solutions and Bäcklund Transformations 39

3.6 The Additional $\mathbb{S}^1$ Action 42

3.7 Harmonic Maps into Grassmannians 43

4 Harmonic Maps into Compact Lie Groups 47

4.1 Symmetry group of the harmonic map equation . . . 48

4.2 A New Formulation 49

4.3 Harmonic Maps into Grassmannian, Another Point of View 53

Bibliography

Chapter 1

Introduction

The concept of harmonic maps is a generalization of the concept of geodesics. The harmonic maps in this paper go from a two-dimensional domain to compact Lie groups; hence they are two-dimensional analogues of geodesics. They encompass many fundamental examples in differential geometry, such as minimal surfaces (in fact, one can show that conformal harmonic maps are minimal surfaces), which have been studied by geometers over a long period of time. Since the late 70s, the field has been acquired new vitality from mathematical physics, in the guise of the non-linear sigma model, which is a harmonic map from a flat Minkowski space-time into a Riemannian manifold. As a result, harmonic maps have attracted the attention of a much wider audience than before, both within the mathematical community and beyond.

One of the themes of research in this area during the last 25 years or so is the classification of such harmonic maps, i.e., the description(or parametrization) of harmonic maps from Riemann surfaces to compact Lie groups or symmetric spaces, in terms of well known maps. There are several reasons for doing this. The most obvious one is that such a description provides the general solution of the relevant harmonic map equation. Another is that such a description should be useful in describing the moduli space of solutions of the harmonic map equation.

The nature of this problem turns out to be algebraic, rather than analytic; it is global than local. The methods used to study the problem are therefore closer to algebra and topology than to analysis.

A turning point in the theory was the idea that the harmonic map equation is a kind of "integrable system". This idea arose explicitly in the mathematical physics literature, and the harmonic map equation was reformulated as a kind of parametrized Lax equation. This was taken up in K. Uhlenbeck [12], where several new results were obtained. Soon the potential of some Lie theoretic methods, by employing certain infinite dimensional Lie algebra (affine *Kac-Moody* Lie algebra) and Lie group(loop group), in the harmonic map problem was suggested in Segal [11].

It is rather difficult to define the term "integrable system" accurately, due to the breath of the subject and the whole range of point of view, from very "pure" to very "applied". But there is one common thread, namely the idea of a symmetry group of a differential equation. The nicest and most natural equations often admit symmetry groups. The existence of a large enough symmetry group leads to the possibility of solving the differential equation by algebraic means, and this is presumably the fundamental property of an integrable system.

Let us briefly review Uhlenbeck's idea and see how the theory of integrable systems applied ([12]): Let $\Omega G = L_2(\mathbb{S}^1, 1; G, e)$ be the infinite-dimensional Banach manifold of based loops on G. Pointwise multiplication of loops endows Ω with the structure of a Banach Lie group with Lie algebra $\Omega\mathbf{g} = L_2(\mathbb{S}^1, 1; \mathbf{g}, e)$. In addition, ΩG has a left-invariant Kählerian complex structure given by the identity:

$$\Omega\mathbf{g} = \{\sum_{n\in\mathbb{Z}} \xi_n(\lambda^{-n} - 1) : \lambda \in \mathbb{S}^1, \xi_n \in \mathbf{g}^{\mathbb{C}}\},$$

$$\Omega^{0,1}\mathbf{g} = \{\xi \in \Omega\mathbf{g} : \xi_n = 0 \text{ for } n \leq 0\}$$

The Kähler form at the identity is

$$S(\xi, \eta) = \int_{\mathbb{S}^1} (\xi', \eta) dt,$$

where ξ, $\eta \in \Omega G$ and the Kähler metric is the $L^2_{\frac{1}{2}}$ metric. It is easy to see that the left invariant 2-form defined by S is closed . Thus S is a symplectic form on ΩG([8]).

Now, let $\phi : \mathbb{S}^2 \to G$ be a harmonic map and denote $\phi^*\theta$ by A, where θ is the Maurer-Cartan form on G. Then the harmonicity of ϕ gives

$$d^* A = 0,$$

while the pull-back of the Maurer-Cartan equations gives

$$dA + \frac{1}{2}[A \wedge A] = 0.$$

Define a family of G-connection of the trivial bundle $\Omega \times \mathbb{C}^n$ by

$$\nabla^\lambda = (\bar{\partial} + (1 - \lambda)A_{\bar{z}}, \partial + (1 - \lambda^{-1})A_z)$$

for $\lambda \in \mathbb{S}^1$. Then (??) and (??) are satisfied if and only if each ∇^λ is flat. In this case, since $\mathbb{S}^2$ is simply-connected, each ∇^λ is gauge-equivalent to the trivial connection d so that we have a family of maps $\phi_\lambda : \mathbb{S}^2 \to G$, $\lambda \in \mathbb{S}^1$, each defined up to a left multiplication by a constant, such that

$$\phi_\lambda^* \theta = \frac{1}{2}(1 - \frac{1}{\lambda})A^{1,0} + \frac{1}{2}(1 - \lambda)A^{0,1}.$$

Choosing the constants so that $\phi_1 \equiv I$, $\phi_{-1} = \phi$, we define a map $\Phi : \mathbb{S}^2 \to \Omega G$ by

$$\Phi(x)(\lambda) = \phi_\lambda(x).$$

If Ξ is the left Maurer-Cartan form of ΩG, then

$$\Phi^* \Xi(\lambda) = \frac{1}{2}(1 - \frac{1}{\lambda})A^{1,0} + \frac{1}{2}(1 - \lambda)A^{0,1},$$

from which we conclude that Φ is holomorphic and *pseudo-horizontal* in the sense that

$$\mathrm{Im}\Phi^*\Xi \subset \mathbf{g}^{\mathbb{C}} \otimes \mathrm{span}\{\lambda^{-1} - 1, \lambda - 1\}.$$

Conversely, if $\Phi : \mathbb{S}^2 \to \Omega G$ is holomorphic and pseudo-horizontal, then $\Phi(-1) : \mathbb{S}^2 \to G$ is harmonic map.

Hence it is natural to associate harmonic maps from the Riemann sphere into a (compact) Lie group G to holomorphic liftings of these maps into the loop space ΩG of G.

More precisely, any harmonic map $f : \mathbb{S}^2 \to U_n$ has a canonical factorization:

$$\begin{tikzcd} & \Omega U_n \arrow[d, "\epsilon"] \\ \mathbb{S}^2 \arrow[ur, "\hat{f}"] \arrow[r, "f"] & U_n \end{tikzcd}$$

where $\hat{f}$ is holomorphic and ϵ is the evaluation map $\gamma \mapsto \gamma(1)$. In fact, there is a one-to-one correspondence between base-point-preserving harmonic maps $\mathbb{S}^2 \to U_n$ and normalized horizontal holomorphic maps $\mathbb{S}^2 \to U_n$.

In general, by a "twistor construction" of a map $f : X \to Y$ of complex manifolds we mean a factorization $f = \pi \cdot g$ through an (almost) complex manifold Z, where $g : X \to Z$ is holomorphic and $\pi : Z \to Y$ is a fibre bundle.

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ X \arrow[ur, "g"] \arrow[r, "f"] & Y \end{tikzcd}$$

We see that in the case of harmonic maps into U_n, the loop group ΩU_n is a good candidate for the twistor space.

The use of twistorial methods allows us, in some cases, to reduce the study of harmonic maps to a problem in algebraic geometry. For example, the loop group ΩU_n carries more algebraic structure than U_n. ΩU_n can be identified with a homogeneous space $\Lambda GL_n(\mathbb{C})/\Lambda_+ GL_n(\mathbb{C})$ through the Iwasawa decomposition

$\Lambda GL_n(\mathbb{C}) = \Omega U_n \Lambda_+ GL_n(\mathbb{C})$. It is natural to define an action of $\Lambda_+ GL_n(\mathbb{C})$ on ΩU_n given simply by $\gamma \cdot \delta = (\gamma\delta)_u$, where $\gamma\delta = (\gamma\delta)_u(\gamma\delta)_+$ is the Iwasawa factorization of the loop group version. This kind of actions is usually called a "*dressing transformation*". That gives an action of the loop group $\Lambda GL_n(\mathbb{C})$ on maps $\mathbb{S}^2 \to \Omega U_n$, i.e., the "extended solutions" in Uhlenbeck's words [12].

Actually, the action used by Uhlenbeck [12] is a little more complicated, which is a variation of the *Birkhoff decomposition*. It reads

$$\Lambda_{\mathbb{R}}^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} = (\Lambda_{E,\mathbb{R}}^{1} GL_n\mathbb{C})(\Lambda_{I,\mathbb{R}} GL_n\mathbb{C}).$$

This gives an action of the holomorphic maps $\mathbb{C}^* \to GL_n(\mathbb{R})$ on the meromorphic maps $\mathbb{A}(\mathbb{S}^2, GL_n(\mathbb{C})$ with the reality restriction.

One can prove that two actions on extended solutions with finite uniton number coincide [5].

The main purpose of this thesis is to give a close look at the treatment of the harmonic map as an integrable system and the development or the exploitation on the symmetry group of harmonic maps into Lie groups and symmetric spaces.

In this thesis, we arrange this topic in three chapters:

In the first chapter, we give some preliminaries about Lie group, Lie algebra and harmonic maps. Most importantly, a brief introduction about some factorization theorems of loop spaces will be given, especially those for the Riemann-Hilbert problem.

In Chapter 2, we review some main results in Uhlenbeck's paper([12]), where Uhlenbeck extensively studied harmonic 2-spheres $\mathbb{S}^2 \to U_n$. (Without loss of generality, we consider the following harmonic map: $s : \Omega \to U_n$, where Ω is a simply-connected domain $\subseteq \mathbb{C} \cong \mathbb{R}^2$, due to the

Theorem 1.1 (Sack-Uhlenbeck). *[10] If $s : \mathbb{R}^2 \to G_{\mathbb{R}}$ is harmonic and*

$$\int_{\mathbb{R}^2} |ds|^2 dx < \infty,$$

then $s : \mathbb{R}^2 = \mathbb{S}^2 - \{\infty\} \to G_{\mathbb{R}}$ extends to a smooth harmonic map $\tilde{s} : \mathbb{S}^2 \to G_{\mathbb{R}}$.)

Several new results were obtained in [12]:

(1) There exists a 1-1 correspondence between harmonic maps $s : \Omega \subseteq \mathbb{C} \to G_{\mathbb{R}} = U_n$ and solutions $E : \mathbb{C}^* \times \Omega \to G = GL_n(\mathbb{C})$, which satisfy the PDE's:

$$\begin{aligned} \bar{\partial} E_\lambda &= (1-\lambda) E_\lambda A_{\bar{z}}, \\ \partial E_\lambda &= (1-\lambda^{-1}) E_\lambda A_z. \end{aligned}$$

Here $E_\lambda = E(\lambda, \cdot)$. In the terminology of Uhlenbeck, E is called an extended harmonic map, or an extended solution.

For every $|\lambda| = 1$, we impose the reality(unitary) condition

$$E_\lambda^{-1} = E_{\bar{\lambda}^{-1}}^*.$$

(In another point of view, we can rewrite the extended solution E as $E : \Omega \to \Omega U_n$, where ΩU_n is the based smooth loop group of U_n. In other words, the harmonic maps are associated to the holomorphic liftings of these maps into the loop groups ΩG of G.)

(2) Uhlenbeck constructed the following group of action:

$$\begin{aligned} \mathbb{A}(\mathbb{S}^2, G) &= \{f : \mathbb{S}^2 - \{p_1, p_2, \cdots, p_l\} \to G \text{ meromorphic} \\ &\qquad \text{with no zero or poles at } (0, \infty) \text{ and } f(1) = I\}. \end{aligned}$$

which acts on a set of extended harmonic maps, imposed the reality condition ($E_{\bar{\lambda}}^* = (E_{\frac{1}{\lambda}})^{-1}$):

$$\begin{aligned}\mathcal{M}^k(G) &= \{E : \mathbb{C}^* \times \Omega \to G \\ &\qquad \text{satisfying the following restrictions.}\} \\ &\qquad (1) \qquad \bar{\partial} E_\lambda = (1-\lambda) E_\lambda A_{\bar{z}}, \quad \partial E_\lambda = (1-\lambda^{-1}) E_\lambda A_z, \\ &\qquad (2) \qquad E_1(q) = I, \\ &\qquad (3) \qquad E_{\bar{\lambda}}^* = (E_{\frac{1}{\lambda}})^{-1}, \\ &\qquad (4) \qquad E_\lambda(q) = \sum_{|\alpha| \le q} E_\alpha(q) \lambda^\alpha.\end{aligned}$$

The action $\#$ is given by $g^\# E = (gE)_1$ or equivalently, $g^\# E = gE(gE)_2^{-1}$, where $(gE)_1$ is the first component in the decomposition of $\Lambda_{\mathbb{R}}^{\epsilon, \frac{1}{\epsilon}} GL_n\mathbb{C} = (\Lambda_{E,\mathbb{R}}^1 GL_n\mathbb{C})(\Lambda_{I,\mathbb{R}} GL_n\mathbb{C})$. Usually, we call the action *dressing transformation.* (Indeed, the map $f \to f^\#$ is a representation.) Hence we have a symmetry group of the extended solutions, which is also a symmetry group of the harmonic map.

(3) Furthermore, Uhlenbeck explicitly gave a Bäcklund transform, with the aid of the "simplest type" element of $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$: The new solution

$$\hat{s} = (\pi - \gamma\pi^\perp) s (\hat{\pi} - \bar{\gamma}\hat{\pi}^\perp), \quad \text{where } \gamma = \frac{1-\bar{\alpha}}{1+\bar{\alpha}} \frac{1+\alpha}{1-\alpha} \in \mathbb{S}^1.$$

can be obtained by solving the ODE's:

$$\begin{aligned}\bar{\partial}\hat{\pi} &= (1-\alpha)\hat{\pi} A_{\bar{z}} - (1-\bar{\alpha}^{-1}) A_{\bar{z}} \hat{\pi} + (\alpha - \bar{\alpha}^{-1}) \hat{\pi} A_{\bar{z}} \hat{\pi}; \\ \partial\pi &= (1-\alpha^{-1})\hat{\pi} A_z - (1-\bar{\alpha}) A_z \hat{\pi} + (\alpha^{-1} - \bar{\alpha}) \hat{\pi} A_z \hat{\pi}.\end{aligned}$$

With these results, some new approach on the study of the harmonic maps into Grassmannians was given.

In Chapter 3, we will give a close look at some symmetry groups of harmonic 2-spheres in compact Lie groups. We will make emphasize on some factorization theorems and symmetric groups, which may works for all compact Lie groups, instead of U_n.

Chapter 2

Preliminary

In this chapter, some basic formulae and properties of Lie groups, Lie algebras and harmonic maps will be given. Some factorization theorems for loop spaces will be reviewed.

2.1 Lie Group and Lie Algebra

Lie Group and Lie Algebra

Definition 2.1. *A differentiable manifold G, which is also a group, is called a Lie group if the differentiable structure is compatible with the group structure. The compatibility here means that the group operations $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ are smooth.*

Definition 2.2. *Let G and H be two Lie groups. A homomorphism of G into H which is also smooth is called a Lie group homomorphism (from G to H). An isomorphism between Lie groups which is also a diffeomorphism is called a Lie group isomorphism.*

Let G be a Lie group. If $\rho \in G$, then the *left translation* $L_\rho : g \mapsto \rho g$ of G onto itself is a diffeomorphism. A vector field X on G is called *left invariant* if

$dL_\rho X = X$ for all $\rho \in G$.

Given a tangent vector $X \in G_e$, there exists a unique left invariant vector field $\hat{X}$ on G such that $\hat{X}_e = X$, which is smooth. In fact, $\hat{X}$ can be defined by

$$(\hat{X}(f))(\rho) = X(f \circ L_{\rho^{-1}}) = (\frac{d}{dt}f(\rho\gamma(t)))_{t=0}$$

for $f \in C^\infty(G)$ and $\rho \in G$, where $\gamma(t)$ is any curve in G with tangent vector X at $\gamma(0) = e$ for $t = 0$. Moreover, if X, $Y \in G_e$, then the vector field $[\hat{X}, \hat{Y}]$ defined by

$$[\hat{X}, \hat{Y}]_\rho(f) = \hat{X}_\rho(Yf) - \hat{Y}_\rho(Xf)$$

is left invariant and $[\hat{X}, \hat{Y}]_e = [X, Y]$. The vector space G_e, with the composition $G_e \times G_e \to G_e$, $(X, Y) \mapsto [X, Y]$(called the Lie bracket), is called the *Lie algebra* of G. We denote it by $\mathbf{g}$.

Example 2.1. *All $n \times n$ non-singular matrices over $\mathbb{R}$, under the matrix multiplication, form a Lie group $GL_n\mathbb{R}$. Its Lie algebra is $\mathbf{gl}_n\mathbb{R}$, the set of all $n \times n$ matrices over $\mathbb{R}$, with the Lie bracket $[A, B] = AB - BA$. Similarly, $GL_n\mathbb{C}$ is a Lie group with the Lie algebra $\mathbf{gl}_n\mathbb{C}$.*

Example 2.2. *Let the unitary group U_n be the closed subgroup of $GL_n\mathbb{C}$ consisting of matrices A satisfying $A^{-1} = \bar{A}^t$. The Lie algebra of U_n is the set of skew-Hermitian matrices with $\bar{B} + B^t = 0$.*

Let V be a vector space over a field K and let $\mathbf{gl(V)}$ denote the vector space of all endomorphisms of V with the bracket $[A, B] = AB - BA$. Then $\mathbf{gl(V)}$ is a Lie algebra (over K). Let $\mathbf{a}$ be a Lie algebra over K. A homomorphism of $\mathbf{a}$ into $\mathbf{gl(V)}$ is called a *representation* of $\mathbf{a}$ on V. In particular, let adX denote the linear transformation $Y \mapsto [X, Y]$ of $\mathbf{a}$. Since $ad([X, Y]) = adXadY - adYadX$, the linear mapping $X \mapsto adX$ $(X \in \mathbf{a})$ is a representation of $\mathbf{a}$ on $\mathbf{a}$. It is called the *adjoint representation* of $\mathbf{a}$ and is denoted by ad.

Group Action, Homogeneous Spaces and Symmetric Spaces

For any element g of a group G, we associate a diffeomorphism D_g of a manifold M in such a way that

$$D_{g_1 g_2} = D_{g_1} \cdot D_{g_2}, \qquad D_{g_1^{-1}} = D_{g_1}^{-1}.$$

Then we say that there is a (left) action of the group G on M.

If G is a Lie group and the map $G \times M \mapsto M$, $(g, x) \mapsto D_g x$, is smooth, then the action is called *smooth*, and M is called a G-manifold. The action D is said to be *transitive* if for any two point x, $y \in M$ there is a $g \in G$ such that $D_g x = y$.

Definition 2.3. *A homogeneous space of a Lie group G is a manifold M with a smooth transitive action D of the Lie group G.*

In particular, a Lie group is a homogeneous space with respect to the left translation $L_\sigma : g \mapsto \sigma g$, where g, $\sigma \in G$.

Let x be any point of a homogeneous space of a Lie group G. The *isotropy group* (or stationary group) H_x of the point x is the stabilizer of x under the action of G:

$$H_x = \{g | D_g(x) = x\}.$$

Lemma 2.1. *All isotropy groups H_x of points x of a homogeneous space are isomorphic to each other.*

Proof. Let x, y be any two points of the homogeneous space and g be an element of the Lie group such that $T_g(x) = y$. Then the map $H_x \to H_y$ defined by $h \mapsto ghg^{-1}$ is an isomorphism. □

Theorem 2.1. *There is a one-to-one correspondence between the points x of a homogeneous space M of a group G and the left coset gH_x of H_x in G, where H_x is the isotropy group of x.*

Proof. Let x_0 be any point of the manifold M. Then for each left coset gH_{x_0}, we associate it with the point $D_g(x_0)$ of M. $\square$

For the sake of convenience, sometimes we denote $D_g x$ by $g \cdot x$.

2.2 Harmonic Maps

Let (M, g) and (N, h) be Riemannian manifolds and $\phi : M \to N$ a smooth map.

Its differential $d\phi$ can be viewed as a section of the bundle $\Lambda(\phi^{-1}TN) = T^*M \otimes \phi^{-1}TN$, and we denote by $|d\phi|$ its norm at a point x of M, induced by the metrics g and h, i.e., the Hilbert-Schmidt norm of the linear map $d\phi(x)$.

If (x^i) and (u^α) are local coordinates around x and $\phi(x)$, we have

$$|d\phi|^2 = g^{ij} h_{\alpha\beta}(\phi)\phi_i^\alpha \phi_j^\beta,$$

where $(\phi_i^\alpha) = (\partial\phi^\alpha / \partial x^i)$ is the local representation of $d\phi$.

We note that $|d\phi|^2$ can be seen as the trace of $\phi^* h$ with respect to g: $|d\phi|^2 = \langle g, \phi^* h\rangle = \text{trace}_g \phi^* h$.

Definition 2.4. *The energy density of ϕ is the function $e(\phi) = \dfrac{1}{2}|d\phi|^2$.*

The energy of ϕ, denoted $E(\phi)$, is given by

$$E(\phi) = \frac{1}{2}\int_M \text{trace}_g \phi^* h \, dvol_M.$$

Definition 2.5. *A smooth map is harmonic if it is a critical point of the energy functional with respect to all compactly supported smooth variations.*

The Euler-Lagrange equation is

$$\tau_\phi \equiv \text{trace}_g \nabla d\phi = 0,$$

where ∇ is the connection on $T^*M \otimes \phi^{-1}TN$ induced by the Levi-Civita connections of M and N. The quantity $\tau_\phi \in C^\infty(\phi^{-1}TN)$ is called the *tension field* of ϕ.

Hence a map ϕ is harmonic if and only if it satisfies the Euler-Lagrange equation $\tau_\phi = 0$.

In particular, if M is a simply connected complex domain with Euclidean metric g and N is a compact Lie group with metric induced by a bi-invariant inner product on its Lie algebra. Then

$$\nabla_{\frac{\partial}{\partial x_i}} d\phi = \frac{\partial}{\partial x_i}(\phi^{-1} d\phi),$$

and hence the Euler-Lagrange equation becomes:

$$g^{ij}(\nabla d\phi)_{ij} = (\nabla d\phi)_{ii} = \frac{\partial}{\partial x}(\phi^{-1}\frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(\phi^{-1}\frac{\partial \phi}{\partial y}) = 0.$$

Definition 2.6. *ϕ is called totally geodesic if $\nabla d\phi = 0$.*

Proposition 2.1. *If (M, g), (N, h) and (P, k) are Riemannian manifolds, $\phi \in \mathcal{C}^\infty(M, N)$ and $\psi \in \mathcal{C}^\infty(N, P)$, then*

$$\nabla d(\psi \circ \phi) = d\psi \circ \nabla d\phi + \nabla d\psi(d\phi, d\phi),$$

$$\tau(\psi \circ \phi) = d\psi \circ \tau(\phi) + \mathrm{trace}\nabla d\psi(d\phi, d\phi).$$

Proof.

$$\begin{aligned}
\nabla d(\psi \circ \phi)(X, Y) &= \nabla_X(d\psi \cdot d\phi \cdot Y) - d(\psi \circ \phi) \cdot \nabla_X Y \\
&= (\nabla_{d\phi \cdot X} d\psi) d\phi \cdot Y + d\psi \cdot \nabla_X(d\phi \cdot Y) - d\psi \cdot d\phi \cdot \nabla_X Y \\
&= \nabla d\psi(d\phi \cdot X, d\phi \cdot Y) + d\psi \cdot \nabla d\phi(X, Y).
\end{aligned}$$

□

In particular, if ϕ and ψ are totally geodesic, so is $\psi \circ \phi$. Also, if ϕ is harmonic and ψ is totally geodesic, then $\psi \circ \phi$ is harmonic.

2.3 Some Factorization theorems

Decomposition of Lie Groups

Definition 2.7. *A Lie algebra is said to be nilpotent if each element X is nilpotent, i.e., $X^k = 0$ for some k. A Lie group is nilpotent if its Lie algebra is nilpotent.*

There is a general decomposition theorem for Lie groups, called the *Iwasawa decomposition.*

Theorem 2.2. *Let G be a compact connected Lie group. Then there exists a decomposition*

$$G^{\mathbb{C}} = GAN, \quad G \cap A = A \cap N = G \cap N = \{e\},$$

where A is abelian and N is nilpotent.

In the case $G = U_n$, by the *Gram-Schmidt Process*, we obtain the decomposition

$$GL_n\mathbb{C} = U_n\Delta_n, \quad U_n \cap \Delta_n = T_n,$$

where Δ_n is the group of upper triangular matrices in $G^{\mathbb{C}}$, and T_n is the group of diagonal matrices in U_n.

Furthermore,

$$\Delta_n = T_n A_n N_n,$$

where A_n is the group of real diagonal matrices with positive entries, and N_n is the group of matrices with all entries in the diagonal equal to 1. Since $T_n \subseteq U_n$, we have $GL_n\mathbb{C} = U_n A_n N_n$, and this gives the Iwasawa decomposition of $GL_n\mathbb{C}$.

Decomposition of Loop Groups and Loop Algebras

If X is a manifold, we denote the free and based loop spaces of X by ΛX and ΩX respectively. In other words,

$$\Lambda X = \{\gamma : \mathbb{S}^1 \to X | \gamma \text{ is smooth } \},$$

$$\Omega X = \{\gamma \in \Lambda X : \gamma(1) = x\},$$

where x is a fixed basepoint of X. If X is a group, we refer to these as the free and based *loop groups* of X, and we take $x = e$, the identity element of the group.

The *Gram-Schmidt decomposition* $GL_n\mathbb{C} = U_n\Delta_N$ can be generalized to a decomposition of the loop group

$$\Lambda GL_n\mathbb{C} = (\Omega U_n)(\Lambda_+ GL_n\mathbb{C}).$$

We describe an important analytical interpretation of the Iwasawa decomposition $\Lambda GL_n\mathbb{C} = (\Omega U_n)(\Lambda_+ GL_n\mathbb{C})$. It depends on another (partial) decomposition, called the *Birkhoff decomposition* or *Birkhoff factorization*.

First we have a Lie algebra decomposition

$$\Lambda \mathbf{gl}_n\mathbb{C} = \Lambda_-^0 \mathbf{gl}_n\mathbb{C} \oplus \Lambda_+ \mathbf{gl}_n\mathbb{C},$$

where

$$\Lambda_+ \mathbf{gl}_n\mathbb{C} = Span\{A_i\lambda^i \mid A_i \in \mathbf{gl}_n\mathbb{C},\ i \geq 0\},$$

$$\Lambda_- \mathbf{gl}_n\mathbb{C} = Span\{A_i\lambda^i \mid A_i \in \mathbf{gl}_n\mathbb{C},\ i \leq 0\},$$

and

$$\Lambda_-^0 \mathbf{gl}_n\mathbb{C} = \{\gamma \in \Lambda_- \mathbf{gl}_n\mathbb{C} \mid \gamma(1) = 0\}.$$

Then we have a partial decomposition:

$$\Lambda GL_n\mathbb{C} \supseteq (\Lambda_-^1 GL_n\mathbb{C})(\Lambda_+ GL_n\mathbb{C}),$$

where $\Lambda GL_n\mathbb{C}$, $\Lambda^1_- GL_n\mathbb{C}$ and $\Lambda_+ GL_n\mathbb{C}$ are similarly defined as their Lie algebraic versions. This is called the *Birkhoff decomposition* of $\Lambda_+ GL_n\mathbb{C}$. Since the intersection $\Lambda^1_- GL_n\mathbb{C} \cap \Lambda_+ GL_n\mathbb{C}$ consists of the loops which extend holomorphically to the entire Riemann sphere, we have

$$\Lambda^1_- GL_n\mathbb{C} \cap \Lambda_+ GL_n\mathbb{C} = \{I\}.$$

Let us now consider the following problem:

Riemann-Hilbert Problem

Let Γ be a simple closed contour in the Riemann sphere $\mathbb{C} \cup \infty$. Let F be a smooth matrix-valued function on Γ. When can we find matrix-valued functions F_+, F_- such that

(1) $F = F_-|_\Gamma F_+|_\Gamma$,

(2) $F_+(F_-)$ is holomorphic on the interior (exterior) of Γ?

Our interest lies on the following case with:

$$\begin{aligned} \Gamma &= C^\epsilon \cup C^{\frac{1}{\epsilon}}, \\ F &= (F^\epsilon, F^{\frac{1}{\epsilon}}) : C^\epsilon \cup C^{\frac{1}{\epsilon}} \to GL_n\mathbb{C}, \text{ is smooth.} \end{aligned}$$

where $C^r = \{\lambda | |\lambda| = r\}$ for $r = \epsilon,\ 1/\epsilon$, with $0 < \epsilon < 1$. Thus the contour consists of two adjoint circles around 0 and ∞.

Let

$$\begin{aligned} \text{interior} &= I = \{\lambda | |\lambda| \le \epsilon\} \cup \{\lambda | \frac{1}{\epsilon} \le |\lambda| \le \infty\} = I^\epsilon \cup I^{\frac{1}{\epsilon}}, \\ \text{exterior} &= E = \{\lambda | \epsilon \le |\lambda| \le \frac{1}{\epsilon}\}. \end{aligned}$$

When does there exist

$$F_I = (F_I^\epsilon, F_I^{\frac{1}{\epsilon}}) : I^\epsilon \cup I^{\frac{1}{\epsilon}} \to GL_n\mathbb{C} \text{ holomorphic, and}$$

$$F_E : E \to GL_n\mathbb{C} \text{ holomorphic}$$

such that $F^\epsilon = F_E F_I^\epsilon$ on C^ϵ and $F^{\frac{1}{\epsilon}} = F_E F_I^{\frac{1}{\epsilon}}$ on $C^{\frac{1}{\epsilon}}$?

This will involves a "partial decomposition", namely

$$\Lambda^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} \supseteq (\Lambda_E^1 GL_n\mathbb{C})(\Lambda_I GL_n\mathbb{C}),$$

where

$$\Lambda^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} = \{\gamma : C^\epsilon \cup C^{\frac{1}{\epsilon}} \to GL_n\mathbb{C} | \gamma \text{ smooth}\} \cong \Lambda GL_n\mathbb{C} \times \Lambda GL_n\mathbb{C},$$

$$\Lambda_E^1 GL_n\mathbb{C} = \{\gamma \in \Lambda^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} | \gamma(1) = Id, \ \gamma \text{ can be extended to } E \text{ holomorphically}\},$$

$$\Lambda_I GL_n\mathbb{C} = \{\gamma \in \Lambda^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} | \gamma \text{ can be extended to } I \text{ holomorphically}\}.$$

This case is interesting because the above partial decomposition is related to the Iwasawa decomposition, and this leads to a "complete answer" to the corresponding Riemann-Hilbert problem.

To see this, we should impose a "reality assumption" $F(1/\bar{\lambda}) = F(\lambda)^{-1*}$ which assures the equivariance with respect to the involutions:

$$\mathbb{S}^2 \to \mathbb{S}^2, \quad \lambda \mapsto \frac{1}{\bar{\lambda}},$$

$$GL_n\mathbb{C} \to GL_n\mathbb{C}, \quad A \mapsto A^{-1*}.$$

We have

Proposition 2.2. *Let F be described as above.*

Assume that

(1) $F = (F^\epsilon, F^{\frac{1}{\epsilon}})$ *satisfies the reality assumption, and*

(2) F^ϵ *extends holomorphically to* $\{\lambda | \epsilon \le |\lambda| \le 1\}$ *(i.e.,* $F^\epsilon = G|_{C^\epsilon}$ *for some holomorphic* $G : \{\lambda | \epsilon \le |\lambda| \le 1\} \to GL_n\mathbb{C}$*).*

Then the required E_E and F_I exist. Moreover, F_E is given by $F_E = G_1$, where $G|_{\mathbb{S}^1} = G_1G_2$ is the factorization of G with respect to the Iwasawa decomposition $\Lambda GL_n\mathbb{C} = (\Omega U_n)(\Lambda_+ GL_n\mathbb{C})$.

Proof. The reality condition is equivalent to the condition

$$F^{\frac{1}{\epsilon}}(\lambda) = F^{\epsilon}\big(\frac{1}{\bar{\lambda}}\big)^{-1*}.$$

Since $G|_{\mathbb{S}^1}$ and G_2 are extended holomorphically to $\{\lambda | \epsilon \leq |\lambda| \leq 1\}$, the same is true for G_1. Since $G_1(\lambda)^* = G_1(\lambda)^{-1}$ for $|\lambda| = 1$, we can further extend G holomorphically to $\{\lambda | \epsilon \leq |\lambda| \leq 1/\epsilon\}$, by defining $G_1(\lambda) = G_1(1/\bar{\lambda})^{-1*}$ for $1 \leq |\lambda| \leq 1/\epsilon$.

We have

$$\begin{aligned} &(F^{\epsilon}(\lambda), F^{\frac{1}{\epsilon}}(\frac{1}{\bar{\lambda}})^{-1*}) \\ = \quad &(G_1(\lambda)|_{C^{\epsilon}}, G_1(\frac{1}{\bar{\lambda}})^{-1*}|_{C^{\frac{1}{\epsilon}}})(G_2(\lambda)|_{C^{\epsilon}}, G_2(\frac{1}{\bar{\lambda}})^{-1*}|_{C^{\frac{1}{\epsilon}}}). \end{aligned}$$

Hence we have

$$F_E = G_1, \quad F_I = (G_2(\lambda)|_{C^{\epsilon}}, G_2(\frac{1}{\bar{\lambda}})^{-1*}|_{C^{\frac{1}{\epsilon}}})$$

and the proof is complete. □

Actually if the condition (2) is dropped, the proposition is still true. We have the following result:

Proposition 2.3 (McIntosh). *([6])*

$$\Lambda_{\mathbb{R}}^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} = (\Lambda_{E,\mathbb{R}}^{\frac{1}{\epsilon}} GL_n\mathbb{C})(\Lambda_{I,\mathbb{R}} GL_n\mathbb{C}),$$

where the suffix $\mathbb{R}$ indicates that the reality condition is in force.

This is a consequence of the fact that the partial decomposition $\Lambda^{\epsilon,\frac{1}{\epsilon}} GL_n\mathbb{C} \supseteq (\Lambda_E^{\frac{1}{\epsilon}} GL_n\mathbb{C})(\Lambda_I GL_n\mathbb{C})$ becomes a decomposition under the reality condition.

Chapter 3

A Survey on Unlenbeck's Results

In [12], Uhlenbeck established a one-to-one correspondence between harmonic maps $s : \Omega \subseteq \mathbb{C} \to G_{\mathbb{R}} = U_n$ and solutions $E : \mathbb{C}^* \times \Omega \to G = GL_n(\mathbb{C})$ to the PDE's :

$$\begin{aligned} \bar{\partial} E_\lambda &= (1-\lambda) E_\lambda A_{\bar{z}}, \\ \partial E_\lambda &= (1-\lambda^{-1}) E_\lambda A_z. \end{aligned}$$

Here $E_\lambda = E(\lambda, \cdot)$, $A_{\bar{z}} = \frac{1}{2} s^{-1} s_{\bar{z}}$ and $A_z = \frac{1}{2} s^{-1} s_z$. In the terminology of Uhlenbeck [12], E is called an extended harmonic map, or an extended solution.

For every $|\lambda| = 1$, with the imposed reality(unitary) condition

$$E_\lambda^{-1} = E_{\bar{\lambda}^{-1}}^*,$$

we can rewrite the extended solution E as a map $E : \Omega \to \Omega U_n$, where ΩU_n is the based smooth loop group of U_n. In other words, the harmonic maps are associated to the holomorphic liftings of these maps into the loop groups ΩG of G.

If we are interested in finding a Bäcklund transform, i.e. a method of obtaining new solutions of a system of PDE's from old solutions via solving ODE's, it is natural to consider an action on a set of extended harmonic maps. In this paper, we will see that there are more algebraic structures than analytic structures built on the loop group. In fact, a loop group enjoys many properties of a finite dimensional generalized flag manifold, which we will discuss in Chapter 3. We can easily construct actions on the different sets of extended solutions which induce symmetry groups of the harmonic maps. As we can see, the simplest situation occurs when the Fourier series of $E(\lambda, z)$ has finitely many terms, i.e., it is algebraic in λ. (The highest power of λ is called a uniton number; the corresponding harmonic maps s are said to have finite uniton number.) Actually, all harmonic maps s which extends to the sphere $\mathbb{S}^2$ are of this kind.

Uhlenbeck [12] constructed the following group of action:

$$\begin{aligned}\mathbb{A}(\mathbb{S}^2, G) &= \{f : \mathbb{S}^2 - \{p_1, p_2, \cdots, p_l\} \to G \text{ meromorphic} \\ &\qquad \text{with no zero or poles at } (0, \infty) \text{ and } f(1) = Id\},\end{aligned}$$

which acts on a set of extended harmonic maps with the reality condition ($E^*_{\bar{\lambda}} = (E_{\frac{1}{\lambda}})^{-1}$):

$$\begin{aligned}\mathcal{M}^k(G) &= \{E : \mathbb{C}^* \times \Omega \to G \text{ satisfying the following restrictions}\}. \\ &(1) \qquad \bar{\partial} E_\lambda = (1-\lambda) E_\lambda A_{\bar{z}}, \quad \partial E_\lambda = (1-\lambda^{-1}) E_\lambda A_z, \\ &(2) \qquad E_1(q) = I, \\ &(3) \qquad E^*_{\bar{\lambda}} = (E_{\frac{1}{\lambda}})^{-1}, \\ &(4) \qquad E_\lambda(q) = \sum_{|\alpha| \le q} E_\alpha(q) \lambda^\alpha.\end{aligned}$$

Indeed, the map $f \to D_f$ is a representation.

Furthermore, Uhlenbeck [12] explicitly gave a Bäcklund transform with the aid of the "simplest type" element of $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$: The new solution

$$\hat{s} = (\pi - \gamma \pi^\perp) s (\hat{\pi} - \bar{\gamma} \hat{\pi}^\perp), \quad \text{where } \gamma = \frac{1-\bar{\alpha}}{1+\bar{\alpha}} \frac{1+\alpha}{1-\alpha} \in \mathbb{S}^1,$$

can be obtained by solving the ODE's:

$$\begin{aligned}\bar{\partial}\hat{\pi} &= (1-\alpha)\hat{\pi} A_{\bar{z}} - (1-\bar{\alpha}^{-1}) A_{\bar{z}} \hat{\pi} + (\alpha - \bar{\alpha}^{-1}) \hat{\pi} A_z \hat{\pi}; \\ \partial \pi &= (1-\alpha^{-1}) \hat{\pi} A_z - (1-\bar{\alpha}) A_z \hat{\pi} + (\alpha^{-1} - \bar{\alpha}) \hat{\pi} A_z \hat{\pi}.\end{aligned}$$

Since we can embed the Grassmannian into the U_n totally geodesically, harmonic maps $\Omega \to Gr_k(\mathbb{C}^n)$ correspond to harmonic maps into U_n. This gives us two ideas: First, we may simplify the study of the harmonic maps into the Grassmannian, once we have full understanding of the harmonic maps into U_n.

Since the Grassmannian is a more "algebraic" object, which is more convenient to study, for any compact connect Lie group G, we may ask if there is any "Grassmannian" model of the loop group ΩG that works like an auxiliary space if we study the general harmonic maps into G. We will discuss this in Chapter 3.

3.1 Preliminary

From the point of view of differential geometry, we are ultimately interested in the harmonic 2-spheres $\phi : \mathbb{S}^2 \to U_n$. However by the result of [10], it is known that if $\phi : \mathbb{R}^2 \to N$ is a harmonic map and if the energy of ϕ is finite, then ϕ has an extension to a harmonic sphere $\tilde{\phi} : \mathbb{S}^2 \to N$.

So we now consider the theory of harmonic maps from a simply-connected complex domain $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ into a complex Lie group $G_{\mathbb{R}}$, which is a real form of a complex group G.

If $s : \Omega \to U_n = G_{\mathbb{R}}$, where $\mathbf{u}(n)$ is equipped with the bi-invariant metric: $\langle u, v\rangle = \text{trace}(uv^*)$ for any $u, v \in \mathbf{u}(n)$, then the energy density is

$$\begin{aligned} e(s) &= \frac{1}{2}\mathrm{Tr}_{g_0}(tr(s^{-1}ds(s^{-1}ds)^*)) \\ &= \frac{1}{2}\mathrm{tr}((s^{-1}s_x)^2 + (s^{-1}s_y)^2). \end{aligned}$$

Then the energy is given by

$$\begin{aligned} E(s) &= \frac{1}{2}\int\int_\Omega (|s^{-1}\frac{\partial s}{\partial x}|^2 + |s^{-1}\frac{\partial s}{\partial y}|^2)dxdy \\ &= -\frac{1}{2}\int\int_\Omega \mathrm{Tr}(\frac{\partial s}{\partial x}\frac{\partial s^{-1}}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial s^{-1}}{\partial y})dxdy. \end{aligned}$$

The Euler-Lagrange equation for the integral E is the equation

$$\frac{\partial}{\partial x}(s^{-1}\frac{\partial s}{\partial x}) + \frac{\partial}{\partial y}(s^{-1}\frac{\partial s}{\partial y}) = 0, \tag{3.1}$$

since the tension field is given by

$$\tau_s = \text{trace}\nabla ds = \frac{\partial}{\partial x}(s^{-1}\frac{\partial s}{\partial x}) + \frac{\partial}{\partial y}(s^{-1}\frac{\partial s}{\partial y}).$$

Let $A_x = \frac{1}{2}s^{-1}\frac{\partial s}{\partial x}$, $A_y = \frac{1}{2}s^{-1}\frac{\partial s}{\partial y}$. We may define the Maurer-Cartan form

$$A = A_x dx + A_y dy = \frac{1}{2}s^{-1}ds.$$

Note that both A_x and A_y are skew-Hermitian matrices. In terms of A, we can write the harmonic equation (3.1) as

$$d^*A = \frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) = 0. \tag{3.2}$$

Also, by the definition of A, we have automatically the identity

$$dA + [A, A] = (\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + 2[A_x, A_y])dx \wedge dy = 0. \tag{3.3}$$

In a simply connected domain, equations (3.2) and (3.3) and a specified based point $s(p)$ at $p \in \Omega$ are equivalent to (3.1). We note that $2(dA + [A, A])$ is the curvature of the connection $d + 2A$ in the trivial bundle $\Omega \times \mathbb{C}^n$. The map $s : \Omega \times \mathbb{C}^n \to \Omega \times \mathbb{C}^n$ is the gauge transformation that transforms the flat connection $d + 2A$ into the trivial connection d. Usually equation (3.3) is called a *zero-curvature equation.*

Since in the later chapters, the holomorphic maps play a fundamental role, we will identify $\mathbb{R}^2$ with $\mathbb{C}$:

$$z = x + iy, \quad \bar{z} = x - iy.$$

Write $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$. The harmonic map equation becomes:

$$\bar{\partial}(s^{-1}\partial s) + \partial(s^{-1}\bar{\partial} s) = 0. \tag{3.4}$$

The 1-form $A = A_z dz + A_{\bar{z}} d\bar{z} = \frac{1}{2}s^{-1}ds$, and the equations (3.2) and (3.3) become

$$d^*A = \partial A_{\bar{z}} + \bar{\partial} A_z = 0, \tag{3.5}$$

$$dA + [A, A] = (\bar{\partial} A_z - \partial A_{\bar{z}} + 2[A_z, A_{\bar{z}}])dz \wedge d\bar{z} = 0. \tag{3.6}$$

In favor of classical integrable system, we may write the above equations as:

$$\bar{\partial} A_z + [A_{\bar{z}}, A_z] = 0, \tag{3.7}$$

$$\partial A_{\bar{z}} + [A_z, A_{\bar{z}}] = 0. \tag{3.8}$$

3.2 Extended Solutions

We will give a reformulation of the zero-curvature equation which provides us better insight:

Theorem 3.1. *Let Ω be simply-connected and $A : \Omega \to T^*(\Omega) \otimes \mathrm{g}$. Then $2A = s^{-1}ds$, where s is harmonic if and only if the curvature of*

$$D_\lambda = (\bar{\partial} + (1 - \lambda)A_{\bar{z}}, \partial + (1 - \lambda^{-1})A_z)$$

vanishes for all $\lambda \in \mathbb{C}^* = \mathbb{C} - 0$.

Proof. Recall that the curvature form F of the connection 1-form $B = (1 - \lambda)A_{\bar{z}}d\bar{z} + (1 - \lambda^{-1})A_z dz = B_1 d\bar{z} + B_2 dz$ is given by

$$F_{\mu\nu} = \frac{\partial B_\nu}{\partial x^\mu} - \frac{\partial B_\mu}{\partial x^\nu} + [B_\mu, B_\nu],$$

where $\mu, \nu = 1, 2$.

Write out the curvature equations and expand in λ:

$$\begin{aligned} & \bar{\partial}(1 - \frac{1}{\lambda})A_z - \partial(1 - \lambda)A_{\bar{z}} + [(1 - \lambda)A_{\bar{z}}, (1 - \frac{1}{\lambda})A_z] \\ = \;& \lambda(\partial A_{\bar{z}} - [A_{\bar{z}}, A_z]) + \lambda^0(\bar{\partial} A_z - \partial A_{\bar{z}} + 2[A_{\bar{z}}, A_Z]) + \lambda^{-1}(-\bar{\partial} A_z - [A_{\bar{z}}, A_z]). \end{aligned}$$

If $A = 1/2s^{-1}ds$ for some harmonic map s, then the curvature vanishes, since the coefficients of λ^α, $\alpha = -1, 0, 1$, are zero by equations (3.6), (3.7) and (3.8). On the other hand, if the curvature vanishes for all λ, then the coefficients vanish identically. Hence equations (3.6), (3.7) and (3.8) hold. Since Ω is simply-connected, these equations are equivalent to the harmonic map equation. $\square$

Since Ω is simply-connected, our next step is to trivialize the connections

$$D_\lambda = (\bar{\partial} + (1-\lambda)A_{\bar{z}}, \partial + (1-\lambda^{-1})A_z).$$

It is equivalent to solving the system of linear equations:

$$\bar{\partial}E_\lambda = (1-\lambda)E_\lambda A_{\bar{z}}, \tag{3.9}$$

$$\partial E_\lambda = (1-\lambda^{-1})E_\lambda A_z, \tag{3.10}$$

for $\lambda \in \mathbb{C}^*$. Since the curvature vanishes, the system is solvable:

$$\begin{aligned}
& \bar{\partial}\partial E_\lambda - \partial\bar{\partial}E_\lambda \\
= \; & \bar{\partial}((1-\lambda^{-1})E_\lambda A_z) - \partial((1-\lambda)E_\lambda A_{\bar{z}}) \\
= \; & (1-\lambda^{-1})((1-\lambda)E_\lambda A_{\bar{z}}A_z - E_\lambda[A_{\bar{z}}, A_z]) - (1-\lambda)((1-\lambda^{-1})E_\lambda A_z A_{\bar{z}} - E_\lambda[A_z, A_{\bar{z}}]) \\
= \; & (1-\lambda^{-1})(1-\lambda)E_\lambda[A_{\bar{z}}, A_z] - ((1-\lambda^{-1}) + (1-\lambda))E_\lambda[A_{\bar{z}}, A_z] \\
= \; & 0.
\end{aligned}$$

Hence the compatibility condition is satisfied and a solution is uniquely determined by prescribing $E_\lambda(p)$ for any fixed point $p \in \Omega$.

Theorem 3.2. *If s is harmonic and $s(p) \equiv I$, then there exists a unique $E : \mathbb{C}^* \to G$ satisfying* (3.9) *and* (3.10) *with*

$$E_1 \equiv I, \quad E_{-1} = s \quad \textit{and} \quad E_\lambda(p) = I.$$

E is analytic and holomorphic in $\lambda \in \mathbb{C}^*$. Moreover, if s is unitary, then E_λ is unitary for $|\lambda| = 1$.

Proof. The existence and uniqueness of E_λ follow from equations (3.9) and (3.9). The regularity follows from the same conditions and the analyticity of s. One can check that

$$\begin{aligned} -\bar{\partial} E_\lambda^{-1} &= (1-\lambda) A_{\bar{z}} E_\lambda^{-1}, \\ -\partial E_\lambda^{-1} &= (1-\lambda^{-1}) A_z E_\lambda^{-1}. \end{aligned}$$

Then taking the adjoint

$$\begin{aligned} -\bar{\partial}(E_\lambda^{-1})^* &= (1-(\lambda^{-1})^*)(E_\lambda^{-1})^*(A_z)^*, \\ -\partial(E_\lambda^{-1})^* &= (1-\bar{\lambda})(E_\lambda^{-1})^*(A_{\bar{z}})^*. \end{aligned}$$

When s is unitary, $(A_z)^* = -A_{\bar{z}}$. If $|\lambda| = 1$, the uniqueness of solution of equation (3.9) and (3.10) gives $(E_\lambda^{-1})^* = E_\lambda$. □

Theorem 3.3. *Suppose $E : \mathbb{C}^* \times \Omega \to G$ is analytic and holomorphic in the first variable, $E_1 \equiv I$ and the expressions*

$$\frac{{E_\lambda}^{-1} \bar{\partial} E_\lambda}{1-\lambda} \text{ and } \frac{{E_\lambda}^{-1} \partial E_\lambda}{1-\frac{1}{\lambda}}$$

are constant in λ. Then $s = E_{-1}$ is harmonic.

Proof. Let

$$\begin{aligned} A &= \frac{1}{2} s^{-1} ds = \frac{{E_{-1}}^{-1} \bar{\partial} E_{-1}}{1-(-1)} d\bar{z} + \frac{{E_{-1}}^{-1} \partial E_{-1}}{1-\frac{1}{-1}} dz \\ &= {E_\lambda}^{-1} \left(\frac{\bar{\partial} E_\lambda}{1-\lambda} d\bar{z} + \frac{\partial E_\lambda}{1-\frac{1}{\lambda}} dz \right). \end{aligned}$$

By comparison, we have

$$\bar{\partial} E_\lambda = (1-\lambda) E_\lambda A_{\bar{z}}, \quad \partial E_\lambda = (1-\lambda^{-1}) E_\lambda A_z.$$

It follows that the connection $(\bar{\partial} + (1-\lambda) A_{\bar{z}}, \partial + (1-\lambda^{-1}) A_z)$ have zero curvature, which implies that $E_{-1} = s$ is harmonic. □

By the above two theorems, we have established a one-to-one correspondence between a harmonic map s and an E_λ. Usually, E_λ is called an *extended solution.*

For the convenience of later discussion, we would like to reformulate the above in terms of loop spaces:

Definition 3.1. *A map $\Phi : \Sigma \to \Omega G$ is an extended solution if it satisfies the equation*

$$\begin{aligned} \Phi^{-1}\Phi_z &= \frac{1}{2}(1 - \frac{1}{\lambda})A \\ \Phi^{-1}\Phi_{\bar{z}} &= \frac{1}{2}(1 - \lambda)B \end{aligned}$$

for some map $A, B : \Sigma \to \mathbf{g}^{\mathbb{C}}$.

The correspondence between harmonic maps and extended solutions is given in the following:

Theorem 3.4. (1) *If $\Phi\Sigma \to \Omega G$ is an extended solution, then the map $\phi : \Sigma \to G$ defined by $\phi(z) = \phi(z, -1)$ is harmonic;*

(2) *If $\phi : M \to G$ is harmonic and M is simply connected, then there exists an extended solution $\phi : M \to G$ such that $\phi(z) = \Phi(z, -1)$. This Φ is unique up to multiplication on the left by an element $\gamma \in \Omega G$ such that $\gamma(-1) = e$.*

Hence in later discussion, we mostly study the equation for E_λ, instead of for s.

The *unitary condition* (or *reality condition*) is the requirement that E_λ be unitary for $|\lambda| = 1$. Note that we may expand E_λ in a Laurent series,

$$E_\lambda = \sum_{\alpha=-\infty}^{\infty} T_\alpha \lambda^\alpha,$$

where $T_\alpha : \Omega \to GL_n\mathbb{C}$. The unitary condition is

$$E_\lambda^{-1} = \sum_{\alpha=-\infty}^{\infty} T_\alpha^* \lambda^{-\alpha}.$$

Usually, we just use the following identity :

$$E_\lambda^{-1} = \sum_{\alpha=-\infty}^{\infty} T_\alpha^*(\lambda^*)^\alpha = \sum_{\alpha=-\infty}^{\infty} (T_\alpha \lambda^\alpha)^* = E_\lambda^*$$

.

3.3 The Variational Formulas for the Extended Solutions

We assume that the solution space for the harmonic equation is a differentiable manifold $\mathcal{M}$ with the tangent space given by the linearized equation.

Definition 3.2. *A conservation law for an equation is a vector field along the solution manifold* $\mathcal{M}$.

The Lie algebra structure of the conservation laws, or the infinitesimal symmetries, is just the Lie bracket of the vector fields. So, to verify the representation, it is necessary to carry out some very messy computations with the Lie brackets. Since we will give a representation of a group in the next section, we can avoid the computation and transform the hidden symmetry into a constructable one.

For every local harmonic map $s : \Omega \to U_n$, we obtain the 1-form $A = \frac{1}{2}s^{-1}ds$. We recall that the second variational formula which is expressed in terms of $\Lambda = s^{-1}\delta s$, where $\Lambda : \Omega \to \mathbf{u}(n)$. We call Λ a Jacobi field if

$$0 = d^*(d\Lambda + 2[A, \Lambda]) = \frac{\partial}{\partial x}(\frac{\partial}{\partial x}\Lambda + 2[A_x, \Lambda]) + \frac{\partial}{\partial y}(\frac{\partial}{\partial y} + 2[A_y, \Lambda]). \quad (3.11)$$

Theorem 3.5. *If* $\lambda \in \mathbb{C}^*$, *then* $\Lambda_\lambda(a) = E_\lambda^{-1} a E_\lambda$ *solves the linearized equation* 3.11. *If* $|\lambda| = 1$ *and* a *is skew, then* $\Lambda_\lambda(a) : \Omega \to \mathbf{u}(n)$ *is also skew.*

Proof. Since E_λ is unitary when $|\lambda| = 1$, the skew-Hermitian condition follows. To verify that Λ satisfies the linearized equation, we write

$$P_{\bar{z}} = (1 - \lambda)A_{\bar{z}} \quad P_z = (1 - \frac{1}{\lambda})A_z$$

and $\Lambda = \Lambda_\lambda(a)$ so that

$$\bar{\partial}\Lambda = [\Lambda, P_{\bar{z}}], \quad \partial\Lambda = [\Lambda, P_z].$$

The equation (3.11) becomes

$$\begin{aligned}
& (\partial\bar{\partial} + \bar{\partial}\partial)\Lambda + 2[A_z, \bar{\partial}\Lambda] + 2[A_{\bar{z}}, \partial\Lambda] \\
= \; & \partial[\Lambda, P_{\bar{z}}] + \bar{\partial}[\Lambda, P_z] + 2[A_z, \bar{\partial}\Lambda] + 2[A_{\bar{z}}, \partial\Lambda] \\
= \; & [2A_{\bar{z}} - P_{\bar{z}}, [\Lambda, P_z]] + [2A_z - P_z, [\Lambda, P_{\bar{z}}]] + [\Lambda, \partial P_{\bar{z}} + \bar{\partial} P_z] \\
= \; & [\Lambda, (1-\lambda)\bar{\partial}A_z + (1 - \frac{1}{\lambda})\partial A_{\bar{z}}] + (\lambda - \frac{1}{\lambda})([A_{\bar{z}}, [\Lambda, A_z]] - [A_z, [\Lambda, A_{\bar{z}}]]) \\
= \; & [\Lambda, (1-\lambda)\bar{\partial}A_z + (1 - \frac{1}{\lambda})\partial A_{\bar{z}} - (\lambda - \frac{1}{\lambda})[A_z, A_{\bar{z}}]].
\end{aligned}$$

This vanishes identically due to the equations (3.7) and (3.8).

□

Now we know that for $\gamma \in \mathbb{C}^*$ and $a \in \mathbf{g}$,

$$\delta s = \delta E_{-1} = s\Lambda_\gamma(a) = E_{-1}E_\gamma^{-1}aE_\gamma$$

gives an infinitesimal variation of the harmonic maps. We know that it will determine δE_λ for $\lambda \in \mathbb{C}^*$ and the expression for δE_λ can be calculated. We actually choose $\delta s = s(E_\gamma^{-1}aE_\gamma - s^{-1}as)$ to preserve the normalization $E_\lambda(p) = I$.

Proposition 3.1. *Given the first variation $\delta s = s(E_\gamma^{-1}aE_\gamma - s^{-1}as)$, we obtain*

$$\delta E_\lambda = \frac{(\lambda-1)(\gamma+1)}{2(\lambda-\gamma)}(E_\lambda E_\gamma^{-1}aE_\gamma - aE_\lambda).$$

Proof.

$$\begin{aligned}
\delta A_{\bar{z}} &= \delta(\frac{1}{2}s^{-1}\bar{\partial}s) \\
&= \frac{1}{2}(-s^{-1}\delta s s^{-1})\bar{\partial}s + \frac{1}{2}s^{-1}\bar{\partial}(sE_\gamma^{-1}aE_\gamma - as) \\
&= \frac{1}{2}\bar{\partial}(E_\gamma^{-1}aE_\gamma) + [A_{\bar{z}}, E_\gamma^{-1}aE_\gamma] \\
&= -\frac{1}{2}(\gamma+1)[E_\gamma^{-1}aE_\gamma, A_{\bar{z}}].
\end{aligned}$$

Hence

$$\bar{\partial}(\delta E_\lambda E_\lambda^{-1}) = (1-\lambda)E_\lambda \delta A_{\bar{z}} E_\lambda^{-1} = -\frac{1}{2}(\gamma+1)(1-\lambda)E_\lambda[E_\gamma^{-1}aE_\gamma, A_{\bar{z}}]E_\lambda^{-1},$$

and

$$\bar{\partial}(E_\lambda E_\gamma^{-1}aE_\gamma E_\lambda^{-1}) = (\lambda-\gamma)E_\lambda[E_\gamma^{-1}aE_\gamma, A_{\bar{z}}]E_\lambda^{-1}.$$

Consequently,

$$\bar{\partial}(\delta E_\lambda E_\lambda^{-1}) = -\frac{1}{2}\frac{(\gamma+1)(1-\lambda)}{\lambda-\gamma}\bar{\partial}(E_\lambda E_\gamma^{-1}aE_\gamma E_\lambda^{-1}).$$

Recall the normalization that $E_\lambda(p) = E_\gamma(p)$ and $\delta E_\lambda(p) = 0$. This gives a candidate formula:

$$\delta E_\lambda E_\lambda^{-1} = -\frac{1}{2}\frac{(\gamma+1)(1-\lambda)}{\lambda-\gamma}((E_\lambda E_\gamma^{-1}aE_\gamma E_\lambda^{-1}) - a).$$

The similar calculation for ∂ will guarantee that that is the only solution.

□

After changing variables to

$$i\tau = \frac{(\lambda-1)}{(\lambda+1)} \quad \text{and} \quad i\xi = \frac{(\gamma-1)}{(\gamma+1)},$$

the singularities now are at τ, $\xi = \pm i$. In these variables $F_0 = I$, $F_\infty = s$ and

$$\delta F_\tau = \tau\frac{(F_\tau F_\xi^{-1}aF_\xi - aF_\tau)}{\tau-\xi}.$$

Fix a harmonic map s, we define the representation on a basis at^α by

$$at^\alpha \mapsto \Lambda^\alpha(a) \triangleq \begin{cases} \frac{1}{\alpha!}(\frac{\partial}{\partial\tau})^\alpha|_{\tau=0}\Lambda_\tau(a) & \text{if} \quad \alpha > 0, \\ -\frac{1}{\alpha!}(\frac{\partial}{\partial\tau})^\alpha|_{\tau=0}\Lambda_{\tau^{-1}}(a) & \text{if} \quad \alpha < 0, \\ a - s^{-1}as & \text{if} \quad \alpha = 0. \end{cases} \tag{3.12}$$

Let f be of finite Fourier series. For $f(\xi) = \sum_{\alpha>0} a_\alpha \xi^\alpha$, we develop the expression in counterclockwise contour integral in $|\xi| < 1$ enclosing 0 to get the appropriate representation

$$\delta F_\tau = \frac{\tau}{2\pi i}\oint_{|\xi|<1}\sum_\alpha \frac{F_\tau F_\xi^{-1}a_\alpha\xi^{-\alpha-1} - a_\alpha\xi^{-\alpha-1}F_\tau}{\tau-\xi}d\xi. \tag{3.13}$$

On the other hand, if $f(\xi) = \sum_{\alpha>0} a_\alpha \xi^{-\alpha}$, we have that in the clockwise contour integral at $\xi = \infty$:

$$\delta F_\tau = \frac{\tau}{2\pi i} \oint_{|\xi|>1} \sum_\alpha \frac{F_\tau F_\xi^{-1} a_\alpha \xi^{\alpha-1} - a_\alpha \xi^{\alpha-1} F_\tau}{\tau - \xi} d\xi.$$

We have

Theorem 3.6. *Let $f(\xi) = \sum_{\alpha=N}^{M} a_\alpha \xi^\alpha$ for $a_\alpha \in \mathbf{g}$. Then the variation of F_τ associated with f is*

$$\delta F_\tau = \frac{\tau}{2\pi i} \oint \frac{F_\tau F_\xi^{-1} f(\xi^{-1}) - f(\xi^{-1}) F_\tau}{\tau - \xi} \frac{1}{\xi} d\xi$$

over any contour enclosing the two point $\pm i$ in a counterclockwise direction and avoiding 0.

Now we write the formula in terms of λ and γ. Let $v(\gamma) = f(\xi^{-1})$.

Theorem 3.7. *If v is holomorphic on $\mathbb{C} \cup \{\infty\} - \{1, -1\}$, it represents the variation*

$$\delta E_\lambda = \frac{\lambda - 1}{2\pi i} \oint \frac{E_\lambda E_\gamma^{-1} v(\gamma) E_\gamma - v(\gamma) E_\lambda}{(\gamma - 1)(\lambda - \gamma)} d\gamma,$$

where the contours enclose 1 and -1. If we deform contours to enclose 0 and ∞, we obtain a simple formula which holds for any v holomorphic in the neighborhood of 0 and ∞:

$$\delta E_\lambda = \frac{\lambda - 1}{2\pi i} \oint \frac{E_\lambda E_\gamma^{-1} v(\gamma) E_\gamma}{(\gamma - 1)(\lambda - \gamma)} d\gamma.$$

3.4 The Representation of $\mathbb{A}(\mathbb{S}^2, G)$ on holomorphic maps $\mathbb{C}^* \to G$

In the previous section, we have proven that we have a large number of conservation laws (i.e., vector fields tangent to the space of harmonic maps). This could be an evident of the existence of a symmetry group of the solution space. In this section, we will construct a group representation which comes from a dressing transformation of a loop group, which we will call *Uhlenbeck's action*, and is a modification of the standard *Birkhoff factorization*.

The two domains in Uhlenbeck's action are

$$S^+ = \mathbb{C}^* = \mathbb{S}^2 - (\{0\} \cup \{\infty\}), \quad S_\epsilon^- = \{\lambda : |\lambda| < \epsilon \text{ or } |\lambda| > \epsilon^{-1}\}.$$

The two classes of functions here are

$$\begin{aligned} X^k &= \{e : \mathbb{C}^* \to G \text{ such that } e \text{ and } e^{-1} \\ &\qquad \text{have Laurent expansions of order } k \text{ and } e(1) = I\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{A}(\mathbb{S}^2, G) &= \{f : \mathbb{S}^2 - \{p_1, p_2, \cdots, p_l\} \to G \text{ meromorphic} \\ &\qquad \text{with no zero or poles at } (0, \infty) \text{ and } f(1) = I\}, \end{aligned}$$

where $\{p_1, p_2, \cdots, p_l\}$ is a discrete finite subset of $\mathbb{S}^2$.

With the reality condition, we have

$$X_{\mathbb{R}}^k = \{e \in X^k ; e(\lambda)^{-1} = e(\sigma(\lambda))^*\}$$

and

$$\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G) = \{f \in \mathbb{A}(\mathbb{S}^2, G) \text{ such that } f(\lambda)^{-1} = f(\sigma(\lambda))^*\}.$$

Now the action is similar to the Birkhoff factorization: We write $f^\#(e) = f \cdot e \cdot R$. $R = R_f$ for f. $R \in \mathbb{A}(\mathbb{S}^2, G)$ and e. $f^\#(e) \in X^k$. In other words, e is holomorphic in $\mathbb{C}^*$. f puts in zeroes and poles on the left, and $R = R_f$ takes them off the right.

Lemma 3.1. *If $f^\#$ can be defined, then there exists a unique $f^\#$ taking the value I at $1 \in \mathbb{C}^*$. Moreover, if $f^\#(e)$ and $g^\#(f^\#(e))$ are defined and normalized to be I at $\lambda = 1$, then $(gf)^\#$ is defined and*

$$(gf)^\#(e) = g^\#(f^\#)(e).$$

This follows from the uniqueness of the factorization (up to the normalization) and Liouville theorem.

Proof. We can always disregard the normalization in this case, since we can always replace $f^{\#}(e)$ and R by $R(f^{\#}(e)(1))^{-1}$ after some factorizations.

Suppose that we have two such normalized factorizations

$$\tilde{f}^{\#}(e)\tilde{R}^{-1} = f^{\#}(e)R^{-1} = fe.$$

Then we have

$$Q = f^{\#}(e)^{-1}\tilde{f}^{\#}(e) = R^{-1}\tilde{R}.$$

The function Q is holomorphic in $\mathbb{C}^*$ and in a neighborhood of $(0,\infty)$. Also $Q(1) = Id$. Therefore Q is holomorphic in the whole $\mathbb{S}^2$ and the constant I due to Liouville's theorem. $\square$

The construction of such a successful group action depend on two facts: one is the existence of the action for all "simplest type" $f \in \mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$. This means that simplest type elements can act on X_k. The other one is the factorization of an arbitrary f into factors of "simplest type".

Definition 3.3. *$f \in \mathbb{A}(\mathbb{S}^2, G)$ is of simplest type if $f(\lambda) = \pi + \xi_\alpha(\lambda)\pi^\perp$, where π is a Hermitian projection onto a complex subspace, $\pi^\perp = 1 - \pi$ is the projection on the orthogonal subspace and $\xi(\lambda)$ is a rational complex function of degree one which is 1 at $\lambda = 1$.*

In addition, if we impose the reality condition on f, then

$$\xi_\alpha(\lambda) = \frac{\lambda - \alpha}{\bar{\alpha}\lambda - 1} \cdot \frac{\bar{\alpha} - 1}{1 - \alpha}.$$

Since $f \in \mathbb{A}(\mathbb{S}^2, G)$, we have $\alpha \neq (0, \infty)$.

Theorem 3.8. *If $f(\lambda) = \pi + \xi_\alpha(\lambda)\pi^\perp$ is of simplest type, then $e^{\#} = f^{\#}(e) = feR_f$ is always defined.*

Proof. Observe that R_f should be of simplest type in order to cancel the poles of f:

$$R = \hat{\pi} + \xi_{\sigma(\alpha)}(\lambda)^* \hat{\pi}^\perp = \tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp,$$

in which $\hat{\pi}$ is the projection on the subspace $\hat{V} = e(\alpha)^*$, where V is the subspace on which π projects.

Explicitly, we have

$$\begin{aligned} feR_f &= (\pi + \xi_\alpha(\lambda)\pi^\perp)e(\tilde{\pi} + \xi_\alpha(\lambda)^{-1}\tilde{\pi}^\perp) \\ &= \pi e\tilde{\pi} + \xi_\alpha(\lambda)\pi^\perp e\tilde{\pi} + \xi_\alpha(\lambda)^{-1}\pi e\tilde{\pi}^\perp + \pi^\perp e\tilde{\pi}^\perp. \end{aligned}$$

In order to have $f^\# \in X^k_{\mathbb{R}}$, we set

$$\pi^\perp e\tilde{\pi} = 0, \quad \pi e\tilde{\pi}^\perp = 0$$

Since we have imposed the reality condition, then $e(\frac{1}{\bar{\alpha}})^* = (e(\alpha))^{-1}$. Then

$$\pi^\perp e_\alpha\tilde{\pi} = 0 \Longrightarrow \tilde{\pi}^* e(\frac{1}{\bar{\alpha}})^*(\pi^\perp)^* = 0 \Longrightarrow \tilde{\pi}e_\alpha^{-1}\pi^\perp = 0.$$

These two equations become

$$\begin{aligned} \langle \tilde{\pi}e_\alpha^{-1}\pi^\perp y, x\rangle &= 0, \\ \langle \pi e_\alpha \tilde{\pi}^\perp x, y\rangle &= 0, \quad \text{for any } x, y. \end{aligned}$$

Both claim that the range of $\tilde{\pi}$ is $\mathcal{R}(\tilde{\pi}) = e_\alpha^*\mathcal{R}(\pi)$ provided $e_\alpha^{-1} = e_\alpha^*$, which gives $\tilde{\pi}$, since $\tilde{\pi}$ is a Hermitian projection.

□

Theorem 3.9. *Every $f \in \mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$ factors into a product of factors of simplest type and trivial factors in the center.*

Proof. Since f is meromorphic, $f(\lambda)$ has a nonzero kernel at a finite number of points $\lambda = \alpha_1,\ \alpha_1,\ \cdots,\ \alpha_m$. By the reality condition, we see that $|\alpha_i| \neq 1$.

Observe that for a factor of simplest type, $\det(\pi + \xi_\alpha(\lambda)\pi^\perp) = \xi_\alpha(\lambda)^m$ where $m = \text{rank } \pi^\perp$. This determinant has a zero of order m at α and a pole of order m at $(\bar{\alpha})^{-1}$ and total degree m. Since we have assumed that all f satisfies the reality condition, zeros and poles come with this pairing.

Suppose that $f(\lambda)$ has a zero (in the matrix group case, it means $f(\lambda)$ has a zero entry) at α (or equivalently a pole at $1/\bar{\alpha}$). Let

$$\hat{f}(\lambda) = \xi_\alpha(\lambda)^n f(\lambda),$$

for $n \geq 0$ have no pole at $\lambda = \alpha$. Since f can have both zeros and poles at some λ, $\hat{f}$ eliminates the potential poles at $\lambda = \alpha$.

The requirement $\hat{f}(\alpha) \neq 0$ determines n, that is, n is the minimal number of orders of poles of $f(\lambda)$. Write the total order of zeros of f at $(\alpha, 1/\bar{\alpha}) = \mathcal{O}(\alpha, 1/\bar{\alpha})$. Then

$$\mathcal{O}(\alpha, \frac{1}{\bar{\alpha}}) \triangleq \text{the order } m \text{ of the zero of } \det\hat{f}(\lambda) \text{ at } \lambda = \alpha.$$

The *total degree* of f is the sum of all $\mathcal{O}(\alpha, 1/\bar{\alpha})$.

Note that two elements f and $\hat{f} = f \cdot \xi_\alpha(\lambda)$ are of the same degree.

We now may prove the theorem by induction on the total degree of f.

(1) If the degree is zero, that is, $f(\lambda)$ has no pole or zero for all λ, and f is meromorphic, then f must be constant. Since $f(1) = I$, $f(\lambda) = I$.

(2) Suppose we can factorize all f with total degree less than m.

(3) Now suppose g satisfies the reality condition and is of degree m. Without loss of generality, choose any zero α of g and write

$$\hat{g}(\lambda) = \xi_\alpha(\lambda)^n g(\lambda)$$

for $n \geq 0$, so that $\hat{g}(\alpha)$ is well-defined and zero only if $n = 0$. Let V be the kernel of $\hat{g}(\lambda)$. If $n = 0$, then V may be the whole $\mathbb{C}^n$. In this trivial case

$g(\lambda) = 0$ and

$$g(\lambda) = \xi_\alpha(\lambda) f(\lambda),$$

where $f(\lambda)$ has degree $m-n$, whose factorization is ensured by the hypothesis.

In the case that V is not all of $\mathbb{C}^n$, define π to be the Hermitian projection on the orthogonal complement of V, so $\pi^\perp$ is the projection on V. Let

$$f(\lambda) = \hat{g}(\lambda)(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp).$$

Then

$$f(\alpha) = \hat{g}(\alpha)\pi + \frac{(\bar{\alpha}\alpha - 1)(\bar{\alpha} - 1)}{-\alpha + 1} g'(\alpha)\pi^\perp$$

is finite. Thus $\det(f(\lambda) = \det\hat{g}(\lambda)((\xi_\alpha(\lambda))^{-s}$, where s is the rank of $\hat{g}(\alpha)$. The total degree of

$$f(\lambda) = g(\lambda)(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp)\xi_\alpha^n$$

is $m - s \geq 0$.

□

Theorem 3.10. *There exists a smooth action of $\mathbb{A}_\mathbb{R}(\mathbb{S}^2, G)$ on $X_\mathbb{R}^k$.*

Proof. Let Γ_ϵ be the contour $|\lambda| = \epsilon$, ϵ^{-1} oriented in the usual way about $(0, \infty)$. Suppose also that T is holomorphic in the punctured neighborhoods of the point $(0, \infty)$. We wish to normalize our solution to be zero at 1. we define

$$T_\infty(\lambda) = -\frac{\lambda - 1}{2\pi i} \oint_{\Gamma(\epsilon)} \frac{T(\xi)}{(\xi - 1)(\xi - \lambda)} d\xi$$

for $\lambda \in \mathbb{C} - \{0\}$, $\epsilon < |\lambda| < \epsilon^{-1}$. We have

$$T_R(\lambda) = \frac{\lambda - 1}{2\pi i} \oint_{\Gamma(\delta)} \frac{T(\xi)}{(\xi - 1)(\xi - \lambda)} d\xi$$

for $|\lambda| < \delta$ or $|\lambda| > \delta^{-1}$. On the punctured neighborhood of 0 and ∞

$$T(\lambda) = T_\infty(\lambda) + T_R(\lambda).$$

Such a decomposition is the linear Riemann-Hilbert problem and the decomposition is unique with $T_\infty(1) = 0$, and $T_R(\lambda)$ is holomorphic in neighborhoods of 0 and ∞. Now assume that we have

$$g(\lambda) = f(\lambda)e(\lambda) = e^{\#}R^{-1}(\lambda)$$

and differentiate both sides

$$(e^{\#})^{-1}\delta e^{\#} - R^{-1}\delta R = (e^{\#})^{-1}\delta g R.$$

If we let $T(\lambda) = (e^{\#}(\lambda))^{-1}\delta g(\lambda)R(\lambda)$, then for the decomposition we have $(e^{\#}(\lambda))^{-1}\delta e^{\#}(\lambda) = T_\infty(\lambda)$. If we write $\delta g(\lambda) = v(\lambda)f(\lambda)e(\lambda)$, we obtain

$$\delta e^{\#}(\lambda) = -\frac{\lambda - 1}{2\pi i}e^{\#}(\lambda)\oint_{\Gamma(\epsilon)} \frac{(e^{\#})^{-1}(\xi)v(\xi)e^{\#}(\xi)}{(\xi - 1)(\xi - \lambda)}d\xi.$$

The formula will agree with our factorization via simplest factors by the uniqueness criteria. This is also agrees with Theorem3.7.

□

3.5 An Action of $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$ on extended solutions and Bäcklund Transformations

Once we understand the action of $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$ on $X^k_{\mathbb{R}}$, we consider the set of extended solutions E_λ which have finite Fourier series in λ, instead of $X^k_{\mathbb{R}}$. Precisely, we define

$$
\begin{aligned}
\mathcal{M}^k(G) &= \{E : \mathbb{C}^* \times \Omega \to G \text{ satisfying the following restrictions}\}.\\
(1) &\qquad \bar{\partial} E_\lambda = (1-\lambda) E_\lambda A_{\bar{z}}, \quad \partial E_\lambda = (1-\lambda^{-1}) E_\lambda A_z,\\
(2) &\qquad E_1(q) = I,\\
(3) &\qquad E^*_{\bar{\lambda}} = (E_{\frac{1}{\lambda}})^{-1},\\
(4) &\qquad E_\lambda(q) = \sum_{|\alpha| \le k} E_\alpha(q) \lambda^\alpha.
\end{aligned}
$$

Hence we have the following:

Theorem 3.11. *Given $f \in \mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$, $f^\# : \mathcal{M}^k(G) \to \mathcal{M}^k(G)$. Moreover, $f \to f^\#$ is a representation.*

Proof. By Theorem 3.3, we need to check that

$$\frac{(f^\# E)_\lambda{}^{-1} \bar{\partial}(f^\# E)_\lambda}{1-\lambda}, \quad \frac{(f^\# E)_\lambda{}^{-1} \partial(f^\# E)_\lambda}{1-\frac{1}{\lambda}}$$

are constant in λ. Note that $f^\# E$ is holomorphic except possibly at $0, \infty$. Then $\dfrac{(f^\# E)_\lambda{}^{-1} \bar{\partial}(f^\# E)_\lambda}{1-\lambda}$ is holomorphic except possibly at 1, ∞ and 1. Due to the normalization $(f^\# E)_1 = I$, we have $(\bar{\partial}(f^\# E))_1 \equiv 0$. Hence $\dfrac{(f^\# E)_\lambda{}^{-1} \bar{\partial}(f^\# E)_\lambda}{1-\lambda}$ is holomorphic near 1. On the other hand, since $f^\# E = fER$, we have

$$
\begin{aligned}
\frac{(f^\# E)_\lambda{}^{-1} \bar{\partial}(f^\# E)_\lambda}{1-\lambda} &= \frac{1}{1-\lambda}(R^{-1}E^{-1})(\bar{\partial} E R + E \bar{\partial} R)\\
&= R^{-1} \frac{E^{-1} \bar{\partial} E}{1-\lambda} R + \frac{1}{1-\lambda} R^{-1} \bar{\partial} R.
\end{aligned}
$$

Since $\dfrac{E^{-1} \bar{\partial} E}{1-\lambda}$ is constant in λ, the above expression is holomorphic in λ near 0 and ∞. By *Liouville theorem*,

$$\frac{(f^\# E)_\lambda{}^{-1} \bar{\partial}(f^\# E)_\lambda}{1-\lambda} = \tilde{A}_{\bar{z}}$$

is constant in λ.

Besides that, the normalization follows, since $(f^\# E)_\lambda(p) = f(\lambda) I f^{-1}(\lambda) = I$. □

Let $\wp$ be the set of Hermitian projections on $\mathbb{C}^n$.

Corollary 3.1. *Let $f = \pi + \xi_\alpha(\lambda)\pi^\perp$, where $\pi \in \wp$. Then*

$$f^{\#}(E_\lambda) = (\pi + \xi_\alpha(\lambda)\pi^\perp)E_\lambda(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp),$$

for some $\tilde{\pi} \in \wp$. Moreover, $\tilde{\pi}(q)$ is the Hermitian projection on the subspace $(E_\lambda(q))^*V$, where $\pi : \mathbb{C}^n \to V$.

One of Uhlenbeck's main results is the existence of Bäcklund transform, which is a method of obtaining new solution of a system of partial differential equation from old solution by solving ordinary differential equations.

We have the following

Theorem 3.12. *Let $s : \Omega \to U_n$ be a harmonic map, and $A = \frac{1}{2}s^{-1}ds$. Then a family of new solutions parameterized by $\alpha \in \mathbb{C}^*$ and $\pi \in \wp$ can be found by solving the consistent pair of ODE's for $\tilde{\pi} : \Omega \to \wp$ with $\tilde{\pi}(p) = \pi$:*

$$\begin{aligned} \bar{\partial}\tilde{\pi} &= (1-\alpha)\tilde{\pi}A_{\bar{z}} - (1-\bar{\alpha}^{-1})A_{\bar{z}}\tilde{\pi} + (\alpha - \bar{\alpha}^{-1})\tilde{\pi}A_{\bar{z}}\tilde{\pi}; \\ \partial\tilde{\pi} &= (1-\alpha^{-1})\tilde{\pi}A_z - (1-\bar{\alpha})A_z\tilde{\pi} + (\alpha^{-1} - \bar{\alpha})\tilde{\pi}A_z\tilde{\pi}. \end{aligned}$$

The new solution can be written as

$$\tilde{s} = (\pi - \gamma\pi^\perp)s(\tilde{\pi} - \bar{\gamma}\tilde{\pi}^\perp), \quad \textit{where } \gamma = \frac{1-\bar{\alpha}}{1+\bar{\alpha}}\frac{1+\alpha}{1-\alpha} \in \mathbb{S}^1.$$

Proof. By Corollary 3.1, the extended solution for the harmonic map $\tilde{s}$ is

$$\tilde{E_\lambda} = (\pi + \xi_\alpha(\lambda)\pi^\perp)E_\lambda(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp).$$

Hence

$$\frac{(\tilde{E})_\lambda{}^{-1}\bar{\partial}(\tilde{E}_\lambda)}{1-\lambda}, \quad \frac{(\tilde{E})_\lambda{}^{-1}\partial(\tilde{E}_\lambda)}{1-\frac{1}{\lambda}} \text{ are constant in } \lambda.$$

More explicitly,

$$\begin{cases} \tilde{A}_{\bar{z}} &= (\tilde{\pi} + \xi_\alpha(\lambda)\tilde{\pi}^\perp)[A_{\bar{z}}(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp) + (1-\lambda)^{-1}\bar{\partial}(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp)], \\ \tilde{A}_z &= (\tilde{\pi} + \xi_\alpha(\lambda)\tilde{\pi}^\perp)[A_z(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp) + (1-\frac{1}{\lambda})^{-1}\partial(\tilde{\pi} + \xi_\alpha^{-1}(\lambda)\tilde{\pi}^\perp)] \end{cases}$$

are constant in λ.

One can check that the coefficients of pole of λ at α and $1/\bar{\alpha}$ vanish as follows: For the first equation, we start from

$$\tilde{\pi}[A_{\bar{z}}\tilde{\pi}^{\perp} + (1-\alpha)^{-1}\bar{\partial}\tilde{\pi}^{\perp}] = 0, \tag{3.14}$$

$$\tilde{\pi}^{\perp}[A_{\bar{z}}\tilde{\pi} + (1-\frac{1}{\bar{\alpha}})^{-1}\bar{\partial}\tilde{\pi}] = 0. \tag{3.15}$$

Multiply (3.14) by $(1-\alpha)$, plus $(1-\bar{\alpha}^{-1})$ (3.15), we have the first equation. For the second equation, we consider

$$\begin{aligned} \tilde{\pi}[A_z\tilde{\pi}^{\perp} + (1-\alpha^{-1})^{-1}\partial\tilde{\pi}^{\perp}] &= 0, \\ \tilde{\pi}^{\perp}[A_z\tilde{\pi} + (1-\bar{\alpha})^{-1}\partial\tilde{\pi}] &= 0. \end{aligned}$$

□

3.6 The Additional $\mathbb{S}^1$ Action

We construct an $\mathbb{C}^*$ action and show that it preserves the reality condition when $|\gamma| = 1$, $\gamma \in \mathbb{C}^*$.

Theorem 3.13. *If $\gamma \in \mathbb{C}^*$ and $E \in \mathcal{M}^k(G)$, then*

$$\gamma^{\#}E_\lambda = E_{\lambda\gamma}E_\gamma^{-1} \in \mathcal{M}^k(G) \text{ for } |\gamma| = 1.$$

Proof. The condition $|\gamma| = 1$ is needed only to verify the reality condition. We need to check the harmonic map condition. It reduces to Theorem 3.3. That is, we need to show that

$$\frac{(\gamma^{\#}E_\lambda)^{-1}\bar{\partial}(\gamma^{\#}E_\lambda)}{1-\lambda} \quad \text{and} \quad \frac{(\gamma^{\#}E_\lambda)^{-1}\partial(\gamma^{\#}E_\lambda)}{1-\frac{1}{\lambda}}$$

are constant in λ. In fact

$$\begin{aligned} \frac{(\gamma^{\#}E_\lambda)^{-1}\bar{\partial}(\gamma^{\#}E_\lambda)}{1-\lambda} &= E_\gamma E_{\lambda\gamma}^{-1}\bar{\partial}(E_{\lambda\gamma}E_\gamma^{-1}) \\ &= \frac{E_\gamma(1-\gamma\lambda)A_{\bar{z}}E_\gamma^{-1} - E_\gamma(1-\gamma)A_{\bar{z}}E_\gamma^{-1}}{1-\lambda} \\ &= \gamma E_\gamma A_{\bar{z}} E_\gamma^{-1}; \end{aligned}$$

$$\begin{aligned}
\frac{\gamma^{\#}E_\lambda{}^{-1}\partial(\gamma^{\#}E_\lambda)}{1-\frac{1}{\lambda}} &= E_\gamma E_{\lambda\gamma}^{-1}\partial(E_{\lambda\gamma}E_\lambda^{-1}) \\
&= \frac{E_\gamma(1-\frac{1}{\lambda\gamma})A_zE_\gamma^{-1} - E_\gamma(1-\frac{1}{\gamma})A_zE_\gamma^{-1}}{1-\frac{1}{\gamma}} \\
&= \frac{1}{\gamma}E_\gamma A_z E_\gamma^{-1}.
\end{aligned}$$

This verifies the harmonic map condition.

□

We would like to see the relationship of the actions by $\mathbb{A}(\mathbb{S}^2, G)$ and $\mathbb{C}^*$.

To avoid confusion, we write f^* for the action of $f \in \mathbb{A}(\mathbb{S}^2, G)$. Let $\gamma^{\#}f(\lambda) = f(\gamma\lambda)$ if $f \in \mathbb{A}(\mathbb{S}^2, G)$, $\gamma \in \mathbb{C}^*$.

Proposition 3.2.

$$\gamma^{\#}(f^*E) = (\gamma^{\#}f)^*(\gamma^{\#}E).$$

Proof. This follows from the uniqueness of the factorization of $f(\gamma\lambda)E_{\gamma\lambda}(q)$. We have

$$\gamma^{\#}(f^*E)(\lambda) = f(\gamma\lambda)E_{\gamma\lambda}(q)S_{\gamma\lambda}(E_{\tilde{\gamma}}(q)), f(\tilde{\gamma}) = (f^*E(q))(\gamma\lambda) = (\gamma^{\#}f)^*(\gamma^{\#}E).$$

□

3.7 Harmonic Maps into Grassmannians

Since we know that the Grassmannians $G_k(\mathbb{C}^n)$ can be totally geodesically embedded in the U_n . We may treat the theory of harmonic maps into Grassmannians, in comparison to the theory of harmonic maps into U_n.

We known that the composition of a harmonic map $u : M \to N$ with a totally geodesic map $v : N \to P$ is harmonic.

Let us consider the embedding

$$i : G_k(\mathbb{C}^n) \to U_n, \quad V \mapsto \pi_V - \pi_{V^\perp},$$

where π_V denotes the Hermitian projection operator onto the k-plane V. Equivalently, we may define $G_k(\mathbb{C}^n) \subset G_{\mathbb{R}} = U_n$ and

$G_k(\mathbb{C}^n) = \{\phi \in U_n : \phi^2 = I$ and the eigenspace corresponding to 1 is k-dimensional$\}$.

Proposition 3.3. *Let $\Omega \xrightarrow{s} G_k(\mathbb{C}^n) \xrightarrow{\phi} U_n$. Then s is harmonic if and only if $\phi \circ s$ is harmonic.*

If we consider the harmonic maps into the Grassmannian, the normalization should be changed. We let $E_\lambda(p) = Q_k(\lambda)$, where

(1) $Q_k(1) = I$,

(2) $Q_k(-1) = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$,

(3) $Q_k(-\lambda)Q_k(-1) = Q_k(\lambda)$.

We have the following $\mathbb{S}^1$-action on the harmonic maps into $G_k(\mathbb{C}^n)$:

Proposition 3.4. *If $E_\lambda(p) = Q_k(\lambda)$, we have for $|\gamma| = 1$*

$$\gamma^* s = E_{-\gamma} E_\gamma^{-1} : \Omega \to G_k(\mathbb{C}^n)$$

harmonic if $s : \Omega \to G_k(\mathbb{C}^n)$ is harmonic.

Proof. We have investigated the $\mathbb{S}^1$ action on the harmonic maps into U_n, which ensures that $\gamma^* s : \Omega \to U_n$ is harmonic. We need to check that $(\gamma^* s)^2 = I$. An extended solution for $\gamma^* s$ is given by

$$(\gamma^* E)_\lambda = E_{\lambda\gamma} E_\gamma^{-1}.$$

Now we want to prove that the extended solution for $\gamma = 1$ and $\gamma = -1$ are the same.

Check the normalization:

$$(-1^* E)_\lambda(p) = E_{-\lambda}(p) E_{-1}^{-1}(p) = Q_k(\lambda) = E_\lambda(p),$$

and the harmonic maps:

$$((-1)^*E)_{-1} = E_1E_{-1}^{-1} = s^{-1} = s = (1^*E)_\lambda.$$

Hence the extended solution for $\gamma = 1$ and $\gamma = -1$ are the same, which means that

$$E_\lambda = ((-1)^*E)_\lambda = E_{-\lambda}E_{-1}^{-1} = E_{-\lambda}s.$$

Therefore we have

$$(\gamma^*s)^2 = (\gamma^*E_{-1})^2 = (E_{-\gamma}E_\gamma^{-1})^2 = (E_{-\gamma}s^{-1}E_{-\gamma}^{-1})^2 = E_{-\gamma}s^{-2}E_{-\gamma}^{-1} = I.$$

□

Now like the treatment of the harmonic maps into U_n, we would like to construct an action of some subgroup of $\mathbb{A}_\mathbb{R}(\mathbb{S}^2, G)$, which acts on the Grassmannian preserving the condition $s = s^{-1}$, instead of the $\mathbb{S}^1$ action.

Corresponding to the harmonic map $s = E_{-1}$, we have

$$(f^\# E)_\lambda = f(\lambda)E_\lambda S_\lambda.$$

Assume that the normalization $E_\lambda(p) = Q(\lambda)$ is chosen. Then the extended solution corresponding to $s^{-1} = E_1E_{-1}^{-1}$ is $\tilde{E}_\lambda = E_{-\lambda}E_{-1}^{-1}$. Since

$$s^{-1} = (-1)^*s \text{ and } \tilde{E}_\lambda = (-1)^*E_\lambda = E_{-\lambda}E_{-1}^{-1},$$

We have

$$(g^\#\tilde{E})_\lambda = g(\lambda)\tilde{E}_\lambda\tilde{S}_\lambda = g(\lambda)E_{-\lambda}E_{-1}^{-1}\tilde{S}_\lambda.$$

Our goal is that $(f^\# E)_\lambda$ should be the extended solution for $\tilde{s} = (f^\# E)_{-1}$ and $(g^\#\tilde{E})_\lambda$ should be the extended solution for $\tilde{s}^{-1} = (g^\#\tilde{E})_{-1}$. This means

$$(g^\#\tilde{E})_\lambda = (f^\#\tilde{E})_{-\lambda}(f^\#\tilde{E})_{-1}^{-1}.$$

In other words, if we consider the following diagrams,

$$\begin{tikzcd} s \arrow[r, "f^{\#}"] \arrow[d, ".^{-1}"] & \tilde{s} \arrow[d, ".^{-1}"] \\ s^{-1} \arrow[r, "g^{\#}"] & \tilde{s}^{-1} \end{tikzcd}$$

we seek the proper restriction on g such that the diagram will commute.

Explicitly,

$$g(\lambda)E_{-\lambda}E_{-1}^{-1}\tilde{S}_\lambda = f(-\lambda)E_{-\lambda}S_{-\lambda}\tilde{s}^{-1}.$$

Let $g(\lambda) = f(-\lambda)$. Then

$$(g^{\#}\tilde{E})_\lambda = f(-\lambda)E_{-\lambda}E_{-1}^{-1}\tilde{S}_\lambda = (f^{\#}E)_{-\lambda}Q,$$

where Q is chosen so that $(f^{\#}E)_{-1}Q = I$, or $Q = (f^{\#}E)_{-1}^{-1} = \tilde{s}^{-1}$, as required.

Since the way we choose the normalization, it is not preserved by the action of $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$. We try to manage to have the action preserves the normalization to have $\mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G)$ act in a unique way.

Theorem 3.14. *Suppose that E_λ is an extended harmonic map into $G_k(\mathbb{C}^n)$ satisfying*

(a) *(The reality condition)* $E_\lambda^* = (E_{\frac{1}{\lambda}})^{-1}$,

(b) $E_1 = I$,

(c) $E_\lambda = E_{-\lambda}E_{-1}^{-1}$.

Then the action of $\{f \in \mathbb{A}_{\mathbb{R}}(\mathbb{S}^2, G); f(-\lambda) = f(\lambda)\}$ on E preserves these conditions.

Proof. We need to check that the condition (c) is preserved by the action. In fact

$$\begin{aligned} (f^{\#}E)_\lambda &= f(\lambda)E_\lambda S_\lambda = f(-\lambda)E_{-\lambda}E_{-1}^{-1}S_\lambda \\ &= f(-\lambda)E_{-\lambda}S_{-\lambda}(f(-1)E_{-1}S_{-1})^{-1} \\ &= (f^{\#}E)_{-\lambda}((f^{\#}E)_{-1})^{-1}. \end{aligned}$$

□

Chapter 4

Harmonic Maps into Compact Lie Groups

Assume that G is a compact Lie group, instead of U_n. Then the following still hold:

A map $\phi : \mathbb{C} \to G$ is harmonic if and only if the Euler-Lagrange equation holds, i.e.,

$$(\phi^{-1}\phi_{\bar{z}})_z + (\phi^{-1}\phi_z)_{\bar{z}} = 0.$$

The harmonic equation is equivalent to the zero-curvature equation with parameter λ:

$$\begin{aligned} \bar{\partial} F_\lambda &= (1-\lambda) F_\lambda A_{\bar{z}}, \\ \partial F_\lambda &= (1-\lambda^{-1}) F_\lambda A_z, \end{aligned}$$

(To avoid confusion with those extended solution into U_n, we use F, instead of E, for the extended solution.)

4.1 Symmetry group of the harmonic map equation

Let F be a solution of the harmonic map equation. We may assume that the equation is defined for all $\lambda \in \mathbb{C}^*$. To be precisely, we may assume that we have $F : \mathbb{C} \times \mathbb{C}^* \to G^{\mathbb{C}}$, holomorphic in the second variable, such that $F(z, 1/\bar{\lambda}) = C(F(z, \lambda))$, where $C : G^{\mathbb{C}} \to G^{\mathbb{C}}$ is the involution corresponding to $c : \mathbf{g} \otimes \mathbb{C} \to \mathbf{g} \otimes \mathbb{C}$. (In the case of $G = U_n$, we have $G^{\mathbb{C}} = GL_n\mathbb{C}$, and $C(A) = A^{-1*}$, $c(X) = -X^*$.)

In particular, we may assume that

$$F : \mathbb{C} \to \Lambda_{E,\mathbb{R}} G^{\mathbb{C}},$$

where $E = \{\lambda | \varepsilon \leq |\lambda| \leq \frac{1}{\varepsilon}\}$ and $\mathbb{R}$ indicates that we impose the reality condition $\gamma(1/\bar{\lambda}) = C(\gamma(\lambda))$.

Let $(\gamma^\varepsilon, \gamma^{1/\varepsilon})$ be any element of $\Lambda_{\mathbb{R}}^{\varepsilon,1/\varepsilon} G^{\mathbb{C}}$. Since we have the decomposition

$$\Lambda_{\mathbb{R}}^{\varepsilon,1/\varepsilon} G^{\mathbb{C}} = \Lambda_{E,\mathbb{R}}^1 G^{\mathbb{C}} \Lambda_{I,\mathbb{R}} G^{\mathbb{C}}.$$

One can define

$$\tilde{F} = (\gamma^\varepsilon, \gamma^{1/\varepsilon}) \cdot F = ((\gamma^\varepsilon, \gamma^{1/\varepsilon})F)_E.$$

Proposition 4.1. *$\tilde{F}$ is a solution of the extended harmonic map equation.*

Proof. First, by definition F extends holomorphically in λ to the region E. Hence the same is true for $\tilde{F}^{-1}\tilde{F}_z$. On the other hand, $\tilde{F} = (\gamma^\varepsilon, \gamma^{1/\varepsilon})FH^{-1}$, where $H = ((\gamma^\varepsilon, \gamma^{1/\varepsilon})F)_I$, we have

$$\begin{aligned} \tilde{F}^{-1}\tilde{F}_z &= HF^{-1}(\gamma^\varepsilon, \gamma^{1/\varepsilon})^{-1}((\gamma^\varepsilon, \gamma^{1/\varepsilon})F_z H^{-1} - (\gamma^\varepsilon, \gamma^{1/\varepsilon})FH^{-1}H_z H^{-1}) \\ &= H^{-1}F^{-1}F_z H^{-1} - H_z H^{-1} \\ &= H^{-1}(A + \frac{1}{\lambda}B)H^{-1} - H_z H^{-1} \end{aligned}$$

This extends holomorphically in λ to the region $I - \{0\}$, with a simple pole at $\lambda = 0$.

It follows that $\tilde{F}^{-1}\tilde{F}_z$ extends holomorphically in λ to $\mathbb{C}^* \cup \{\infty\}$, with a simple pole at $\lambda = 0$. This means that it is linear in $1/\lambda$. By the reality condition, we have that $\tilde{F}^{-1}\tilde{F}_{\bar{z}}$ is linear in λ. □

Hence we find a symmetry group of the harmonic map equation, namely $\Lambda_{\mathbb{R}}^{\varepsilon,1/\varepsilon}G^{\mathbb{C}}$.

Theorem 4.1. *The group $\Lambda_{\mathbb{R}}^{\varepsilon,1/\varepsilon}G^{\mathbb{C}}$ acts on the set of $\Lambda^1_{E,\mathbb{R}}G^{\mathbb{C}}$-valued extended harmonic maps.*

4.2 A New Formulation

As we saw in Chapter 3, harmonic maps into U_n is closely related to harmonic maps into the complex Grassmannian. Hence we may ask: For any compact Lie group G, is there any " Grassmannian model" of G such that it may make the study of harmonic maps into G more tractable by studying the harmonic maps into the Grassmannian model instead?

The answer is confirmative, and it is first pointed out by Segal([11]).

Now we introduce some flag manifold such that

$$Gr^{(n)} \cong \Lambda U_n/U_n \cong \Lambda GL_n\mathbb{C}/\Lambda_+GL_n\mathbb{C}. \tag{4.1}$$

Let $H^{(n)}$ be the Hilbert space $L^2(\mathbb{C}^1, \mathbb{C}^n)$. If $e_1, \cdots, e_n$ denote the standard basis vectors for $\mathbb{C}^n$, then the functions

$$\lambda \mapsto \lambda^i e_j, \quad i \in \mathbb{Z}, j = 1, \cdots, n$$

are a basis for $H^{(n)}$. In other words,

$$H^{(n)} = Span\{\lambda \mapsto \lambda^i e_j | i \in \mathbb{Z}, j = 1, \cdots, n\}.$$

Those functions with $i \geq 0$ generate a closed subspace

$$H_+^{(n)} = Span\{\lambda \mapsto \lambda^i e_j | i \geq 0, j = 1, \cdots, n\}$$

of $H^{(n)}$. Let Grass$(H^{(n)})$ denote the space of all vector subspaces $W \subseteq H^{(n)}$ such that

(1) W is closed,

(2) the projection map $W \to H_+^{(n)}$ is Fredholm, and the projection map $W \to (H_+^{(n)})^\perp$ is Hilbert-Schmidt,

(3) the image of the projection $W^\perp \to H_+^{(n)}$, $W \to H_+^{(n)}$ are contained in $C^\infty(\mathbb{S}^1, \mathbb{C}^n)$.

The definition of $Gr^{(n)}$ is as follows.

Definition 4.1.

$$Gr^{(n)} = \{W \in \text{Grass}(H^{(n)}) | \lambda W \subseteq W\}. \tag{4.2}$$

One of the basic results of [9] is

Theorem 4.2.

$$Gr^{(n)} \cong \Lambda U_n / U_n \cong \Lambda GL_n\mathbb{C} / \Lambda_+ GL_n\mathbb{C}.$$

Hence we have $Gr^{(n)} \cong \Omega U_n$.

Let $W : \mathbb{C} \to \Omega U_n$, under the identification $\Omega U_n \cong Gr^{(n)}$. Thus, $W(z) = F(z) H_+^{(n)}$. Consider the following condition

$$\begin{aligned} W_z &\subseteq \tfrac{1}{\lambda} W, \\ W_{\bar{z}} &\subseteq W. \end{aligned} \tag{4.3}$$

Proposition 4.2. *F is an extended solution if and only if W is a solution of equations* (4.3).

Proof. If F is an extended solution, then it is obvious that W is a solution of equations (4.3). Conversely, assume that $W = FH_+^{(n)}$ satisfies equations (4.3). Then we have

$$F^{-1}F_z H_+^{(n)} \subseteq \frac{1}{\lambda} H_+^{(n)}, \quad F^{-1}F_{\bar{z}} H_+^{(n)} \subseteq H_+^{(n)}.$$

From the first of these, we have

$$F^{-1}F_z = \sum_{i \geq -1} A_i \lambda^i.$$

From the second, we have

$$F^{-1}F_{\bar{z}} = \sum_{i \geq 0} B_i \lambda^i.$$

Applying the transformation $X \mapsto -X^*$, we obtain

$$\begin{aligned} F^{-1}F_z &= \sum_{i \geq 0} C_i \lambda^{-i}, \\ F^{-1}F_{\bar{z}} &= \sum_{i \geq -1} D_i \lambda^{-i}. \end{aligned}$$

Combining all the four expressions, we find that $F^{-1}F_z$ is linear in λ^{-1} and $F^{-1}F_{\bar{z}}$ is linear in λ.

□

For any $\gamma \in \Lambda GL_n\mathbb{C}$ and any solution W of the system equations (4.3), we define

$$\gamma \cdot W = \gamma W (= \gamma F H_+^{(n)}),$$

i.e., we use the natural action of $\Lambda GL_n\mathbb{C}$ on $Gr^{(n)} \cong \Lambda GL_n\mathbb{C}/\Lambda_+ GL_n\mathbb{C}$.

Via the identification $\Omega U_n \cong Gr^{(n)}$, this action of $\Lambda GL_n\mathbb{C}$ on ΩU_n is simply given by $\gamma \cdot \delta = (\gamma\delta)_u$, where the factorization $\gamma\delta = (\gamma\delta)_u(\gamma\delta)_+$ is the factorization with respect to the decomposition $\Lambda GL_n\mathbb{C} = (\Omega U_n)(\Lambda_+ GL_n\mathbb{C})$. Hence, γW :

$\mathbb{C} \to Gr^{(n)}$ corresponds to the map $(\gamma F)_u : \mathbb{C} \to \Omega U_n$. We can therefore define our action in the following equivalent way:

$$\gamma \cdot F = (\gamma F)_u.$$

(To avoid confusion, we express the old action in terms of F, and the new action in terms of W.)

Proposition 4.3. *$\gamma \cdot W$ is also a solution of equations* (4.3).

Proof. $\Lambda GL_n\mathbb{C}$ acts linearly on $H_+^{(n)}$ and commutes with multiplication by λ^{-1}. □

Let us consider the relationship between the new action of $\Lambda GL_n\mathbb{C}$ and the old action of $\Lambda_{\mathbb{R}}^{\epsilon,\frac{1}{\epsilon}} GL_n$.

Let $\gamma \in \Lambda_+ GL_n\mathbb{C}$. Then γ extends holomorphically to $\{\lambda \| \lambda| \leq 1\}$. Let us choose such an extension and call it Γ. We construct an element $\tilde{\gamma}$ of $\Lambda_{I,\mathbb{R}} GL_n\mathbb{C}$ by

$$\tilde{\gamma}(\lambda) = \begin{cases} \Gamma(\lambda) & \text{for} \quad |\lambda| = \epsilon \\ \Gamma(\frac{1}{\lambda})^{(-1)*} & \text{for} \quad |\lambda| = \dfrac{1}{\epsilon} \end{cases} \tag{4.4}$$

Theorem 4.3. *If W corresponds to F, then $\gamma \cdot W$ corresponds to $\tilde{\gamma} \cdot F$.*

Proof. To find $\tilde{\gamma} \cdot F$, we must find the "$E - I$" factorization of $\tilde{\gamma} F(z) : C^{\epsilon} \cup C^{\frac{1}{\epsilon}} \to GL_n\mathbb{C}$. Since $\tilde{\gamma} F(z)$ is defined for all λ with $\epsilon \leq |\lambda| \leq 1$, by proposition(2.2), $\tilde{\gamma} F(z)$ is given by the first factor of

$$\tilde{\gamma} F(z)|_{|\lambda|=1}$$

with respect to the decomposition $\Lambda GL_n\mathbb{C} = (\Omega U_n)(\Lambda_+ GL_n\mathbb{C})$. This is just $(\gamma F)_u$. Hence this corresponds to $\gamma \cdot W$. □

4.3 Harmonic Maps into Grassmannian, Another Point of View

Definition 4.2. *A symmetric space is a homogeneous space G/K such that there is an involution $\sigma : G \to G$ (i.e., an automorphism with $\sigma : G \to G$ with order 2) with the following property:*

$$(G_\sigma)_0 \subseteq K \subseteq G_\sigma,$$

where $(G_\sigma)_0$ is the identity component of $G_\sigma = \{g \in G | \sigma(g) = g\}$.

The map $i : G/K \to G$ given by $gK \mapsto \sigma(g)g^{-1}$ defines an immersion of G/K into G. Usually it is called the *Cartan immersion*.

Let $G = U_n$, $K = U_k \times U_{n-k}$. Let

$$\sigma(X) = E_k X E_k^{-1}, \text{ where } E_k = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}.$$

Here I_k denotes the $k \times k$ identity matrix. The symmetric space is the Grassmanian $G_k(\mathbb{C}^n)$. In this example, we have $(G_\sigma)_0 = K = G_\sigma$. Let f be a holomorphic map $\mathbb{C} \to Gr_k(\mathbb{C}^n)$ into the Grassmannian, i the Cartan immersion which is given by $V \mapsto \pi_V - \pi_{V^\perp}$, where $\pi_V : \mathbb{C}^n \to \mathbb{C}^n$ denotes the Hermitian projection operator onto the k-plane V.

Proposition 4.4. *$\phi : \mathbb{C} \to G/K$ is harmonic if and only if $i \circ \phi : \mathbb{C} \to G$ is harmonic.*

Let us consider

$$F : \mathbb{C} \to \Omega U_n : \quad F(z, \lambda) = \pi_{f(z)} + \lambda \pi_{f(z)^\perp}.$$

We will see that F is an extended solution and the harmonic map is given by $\phi(z) = i(f(z)) = F(z, -1)$.

Consider

$$\begin{aligned} F^{-1}F_z &= (\pi_f + \frac{1}{\lambda}\pi_{f^\perp})\frac{\partial}{\partial z}(\pi_f + \lambda\pi_{f^\perp}) \\ &= (\pi_f + \frac{1}{\lambda}\pi_{f^\perp})[\frac{\partial}{\partial z}\circ(\pi_f + \lambda\pi_{f^\perp}) - (\pi_f + \lambda\pi_{f^\perp})\circ\frac{\partial}{\partial z}] \end{aligned}$$

by using the identity

$$\frac{\partial}{\partial z}X = \frac{\partial}{\partial z}\circ X - X\circ\frac{\partial}{\partial z},$$

where X is complex matrix valued and $\partial/\partial z$, X on the right hand side are considered as operators on the functions $\mathbb{C}\to\mathbb{C}^n$, we have,

$$\begin{aligned} F^{-1}F_z &= (\pi_f + \frac{1}{\lambda}\pi_{f^\perp})\circ\frac{\partial}{\partial z}\circ(\pi_f + \lambda\pi_{f^\perp}) - \frac{\partial}{\partial z} \\ &= (\frac{1}{\lambda} - 1)\pi_{f^\perp}\circ\frac{\partial}{\partial z}\circ\pi_f + (\lambda - 1)\pi_f\circ\frac{\partial}{\partial z}\pi_{f^\perp}. \end{aligned}$$

We claim that $\pi_f\circ\frac{\partial}{\partial z}\circ\pi_{f^\perp} = 0$.

Proof. To see this, we use the well known identification

$$\begin{aligned} TGr_k(\mathbb{C}^n)\otimes\mathbb{C} &= T_{1,0}Gr_k(\mathbb{C}^n)\oplus T_{0,1}Gr_k(\mathbb{C}^n) \\ &= Hom(H, H^\perp)\oplus Hom(H^\perp, H), \end{aligned}$$

where H is the tautological vector bundle on $Gr_k(\mathbb{C}^n)$. Since f is holomorphic, the $(0,1)$-part of $Df(\partial/\partial z)$ is zero. Now

$$Df(\frac{\partial}{\partial z}) = \pi_{f^\perp}\circ\frac{\partial}{\partial z}\oplus\pi_f\circ\frac{\partial}{\partial z}\pi_{f^\perp}.$$

So $\pi_f\circ\frac{\partial}{\partial z}\circ\pi_{f^\perp} = 0$. □

The corresponding solution W of

$$\begin{aligned} W_z &\subseteq \tfrac{1}{\lambda}W, \\ W_{\bar z} &\subseteq W \end{aligned}$$

is given by

$$W(z) = F(z,\lambda)H_+^{(n)} = f(z)\oplus\lambda H_+^{(n)}.$$

This new action makes it easy to obtain some new results:

Proposition 4.5. *For $X \in GL_n\mathbb{C}$, $X \cdot (\pi_f + \lambda\pi_{f^\perp}) = \pi_{Xf} + \lambda\pi_{Xf^\perp}$.*

Proof.

$$X(f(z) \oplus \lambda H_+^{(n)}) = Xf(z) \oplus \lambda H_+^{(n)}.$$

□

Hence we have the theorem which claim that the action of the group $\Lambda GL_n\mathbb{C}$ will collapse to a smaller group action:

Theorem 4.4. *For the subgroup $\Lambda_+' GL_n\mathbb{C}$, we have the action $\gamma \cdot (\pi_f + \lambda\pi_{f^\perp}) = \pi_f + \lambda\pi_{f^\perp}$, where $\Lambda_+' GL_n\mathbb{C} = \{\gamma \in \Lambda_+ GL_n\mathbb{C} | \gamma(\lambda) = I + \sum_{i\geq 1} A_i\lambda^i\}$.*

Proof.

$$(I + \sum_{i\geq 1} A_i\lambda^i)(f(z) \oplus \lambda H_+^{(n)}) = f(z) \oplus \lambda H_+^{(n)}.$$

We conclude that the orbit of $\pi_f + \lambda\pi_{f^\perp}$ under $\Lambda GL_n\mathbb{C}$ is given by the orbit of f under $GL_n\mathbb{C} \cong \Lambda_+ GL_n\mathbb{C}/\Lambda_+' GL_n\mathbb{C}$. □

Bibliography

[1] D.V. Alekseevskij, A.M,Vinogradov and V. V. Lychagin, *Geometry I: Basic Ideads and Concepts of Differential Geometry*, Encyclopaedia of Mathematical Sciences, Vol. 28, Springer-Verlag, 1980.

[2] F.E.Burstall, J.H. Raawnsley, *Twistor Theory for Riemannian Symmetric Spaces*, Lecture Notes in Mathematics, Vol. 1424, Springer-Verlag, 1990.

[3] F.E.Burstall, *Recent Developments in Twistor Methods for Harmonic Maps*, Harmonic Mappings, Twistors, and σ-Models, Paul Gauduchow ed., Advanced Series in Mathematical Physics, Vol. 4, World Scientific, 1988.

[4] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, CBMS 50, AMS, 1983.

[5] M. A. Guest and Y. Ohnita, *Group Actions and Deformations for Harmonic Maps*, J. Math. Soc. Japan, Vol. 45, 1993.

[6] S. Helgason, *Differential Geometry and Symmetric Spaces*, Pure and Applied Mathematics, Vol. 12, Academic Press, 1962.

[7] I. McIntosh, *Gobal Solutions of the Elliptic 2D Periodic Toda Lattice*, Nonlinearity, Vol. 7, 1994.

[8] A. N. Pressley, *The energy flow on the loop space of a compact Lie group*, J. London Math. Soc., Vol 26, 1982.

[9] A. N. Pressley and G. B. Segal, *Loop group*, Oxford Univ. Press, 1986.

[10] J. Sacks and K. Uhlenbeck, *The Existence of Minimal Immersion of 2-Spheres*, Ann. of Math. (2) **113**, 1981.

[11] G. Segal, *Loop groups and Harmonic maps*, Advances in Homotopy Theory, London Math. Soc. Lecture Notes 139 , Cambridge University Press, 1989.

[12] K. Uhlenbeck, *Harmonic Maps into Lie Groups(Classical solutions of the chiral model)*, J. Differential Geometry (30) , 1989.